Quiescent Big Bang formation in polarized $U(1)$-symmetry

Liam Urban

January 23, 2025

Der Wissenschaftsfonds.

Kasner spacetimes

Kasner spacetimes

$$M_{\mathsf{Kas}} = (0, \infty) \times \mathbb{T}^3, \quad h_{\mathsf{Kas}} = -dt \otimes dt + \sum_{i=1}^{3} t^{2q_i}\, dx^i \otimes dx^i$$

$$\sum_{i=1}^{3} q_i = \sum_{i=1}^{3} q_i^2 = 1$$

Without loss of generality, $q_1 \leq 0 \leq q_2 \leq \frac{2}{3} \leq q_3 \leq 1$.

Kasner spacetimes

Kasner spacetimes

$$M_{\mathsf{Kas}} = (0, \infty) \times \mathbb{T}^3, \quad h_{\mathsf{Kas}} = -dt \otimes dt + \sum_{i=1}^{3} t^{2q_i}\, dx^i \otimes dx^i$$

$$\sum_{i=1}^{3} q_i = \sum_{i=1}^{3} q_i^2 = 1$$

Without loss of generality, $q_1 \leq 0 \leq q_2 \leq \frac{2}{3} \leq q_3 \leq 1$.

Big Bang

For $q_1 < 0$,

$$\mathcal{K}[h_{\mathsf{Kas}}] = \mathsf{Riem}[h_{\mathsf{Kas}}]_{\alpha\beta\gamma\delta}\mathsf{Riem}[h_{\mathsf{Kas}}]^{\alpha\beta\gamma\delta} \simeq t^{-4}$$

The BKL ansatz

BKL ansatz

For $\overline{M} = (0, \infty) \times N$ with a closed orientable 3-manifold N, and a covector frame $\{\omega_I\}$ on N,

$$h = -dt \otimes dt + \sum_{I=1}^{3} t^{2q_I(x)} \omega_I(x) \otimes \omega_I(x) + \dots$$

$$\sum_{I=1}^{3} q_I(x) = \sum_{I=1}^{3} q_I(x)^2 = 1$$

The BKL ansatz

BKL ansatz

For $\overline{M} = (0, \infty) \times N$ with a closed orientable 3-manifold N, and a covector frame $\{\omega_I\}$ on N,

$$h = -dt \otimes dt + \sum_{I=1}^{3} t^{2q_I(x)} \omega_I(x) \otimes \omega_I(x) + \ldots$$

$$\sum_{I=1}^{3} q_I(x) = \sum_{I=1}^{3} q_I(x)^2 = 1$$

This ansatz is consistent with the Einstein equations if, for $q_1(x) < 0$, it satisfies the integrability condition

$$(\omega_1 \wedge d\omega_1)_x = 0 \quad \Leftrightarrow \exists\, u, v : N \to \mathbb{R} \,:\, \omega_1 = u\, dv$$

The BKL ansatz

BKL ansatz

For $\overline{M} = (0, \infty) \times N$ with a closed orientable 3-manifold N, and a covector frame $\{\omega_I\}$ on N,

$$h = -dt \otimes dt + \sum_{I=1}^{3} t^{2q_I(x)} \omega_I(x) \otimes \omega_I(x) + \dots$$

$$\sum_{I=1}^{3} q_I(x) = \sum_{I=1}^{3} q_I(x)^2 = 1$$

This ansatz is consistent with the Einstein equations if, for $q_1(x) < 0$, it satisfies the integrability condition

$$(\omega_1 \wedge d\omega_1)_x = 0 \quad \Leftrightarrow \exists\, u, v : N \to \mathbb{R} \, : \, \omega_1 = u\, dv$$

$$\Leftrightarrow \quad V_p = \{Y \in T_p N \mid (\omega_1)_p(Y) = 0\} \text{ is integrable}$$

Stable Big Bang formation for Kasner spacetimes

Polarized $U(1)$-symmetric spacetimes

The spacetime admits a non-degenerate, hypersurface-orthogonal spacelike Killing field $\mathcal{X}$.

Stable Big Bang formation for Kasner spacetimes

Polarized $U(1)$-symmetric spacetimes

The spacetime admits a non-degenerate, hypersurface-orthogonal spacelike Killing field $\mathcal{X}$.
Assume M can be foliated by constant time surfaces diffeomorphic to $\Sigma \times \mathbb{S}^1$, and consider (appropriately transported) spatial coordinates $\{x^i\}_{i=1,2,3}$ with $\mathcal{X} = \partial_{x^3}$. Considering the first and second fundamental form $\check{g}$ and $\check{k}$ with respect to this foliation, this is equivalent to $\check{g}, \check{k}$ being independent of x^3 and $\check{g}_{13} = \check{g}_{23} = \check{k}_{13} = \check{k}_{23} \equiv 0$.

Stable Big Bang formation for Kasner spacetimes

Polarized $U(1)$-symmetric spacetimes

The spacetime admits a non-degenerate, hypersurface-orthogonal spacelike Killing field $\mathcal{X}$.
Assume M can be foliated by constant time surfaces diffeomorphic to $\Sigma \times \mathbb{S}^1$, and consider (appropriately transported) spatial coordinates $\{x^i\}_{i=1,2,3}$ with $\mathcal{X} = \partial_{x^3}$. Considering the first and second fundamental form $\check{g}$ and $\check{k}$ with respect to this foliation, this is equivalent to $\check{g}, \check{k}$ being independent of x^3 and $\check{g}_{13} = \check{g}_{23} = \check{k}_{13} = \check{k}_{23} \equiv 0$.

Fournodavlos-Rodnianski-Speck '23

Kasner spacetimes with exponents $(q_1, q_2, q_3) \neq (0, 0, 1)$ exhibit stable Big Bang formation within the class of polarized $U(1)$-symmetric solutions to the Einstein vacuum equations.

Generalised Kasner spacetimes

Einstein scalar-field matter

$$T^{SF}_{\mu\nu} = \overline{\nabla}_\mu \phi \, \overline{\nabla}_\nu \phi + \frac{1}{2} \, \bar{g}_{\mu\nu} \, \overline{\nabla}^\alpha \phi \overline{\nabla}_\alpha \phi$$

$$\mathsf{Ric}[h]_{\mu\nu} = 8\,\pi\, T^{SF}_{\mu\nu}, \quad \Box_g \phi = 0$$

Generalized Kasner spacetimes

$$\overline{M}_{\mathsf{Kas}} = (0,\infty) \times \mathbb{T}^3, \quad h_{\mathsf{Kas}} = -dt \otimes dt + \sum_{i=1}^{d} t^{2q_i} \, dx^i \otimes dx^i$$

$$\phi(t,x) = A \, \log(t), \quad \sum_{i=1}^{3} q_i = \sum_{i=1}^{3} q_i^2 + 8\,\pi\, A^2 = 1$$

For $A \neq 0$, there are solutions with only positive exponents (e.g., FLRW solutions).

The BKL ansatz

BKL ansatz

For $\overline{M} = I \times M^{(3)}$ and a covector frame $\{\omega_I\}$ on M,

$$h = -dt \otimes dt + \sum_{I=1}^{3} t^{2q_I(x)} \omega_I(x) \otimes \omega_I(x) + \dots$$

$$\phi(x) = A(x) \log(t) + \dots$$

$$\sum_{I=1}^{3} q_I(x) = \sum_{I=1}^{3} q_I(x)^2 + 8\,\pi\, A(x)^2 = 1$$

The BKL ansatz

BKL ansatz

For $\overline{M} = I \times M^{(3)}$ and a covector frame $\{\omega_I\}$ on M,

$$h = -dt \otimes dt + \sum_{I=1}^{3} t^{2q_I(x)} \omega_I(x) \otimes \omega_I(x) + \dots$$

$$\phi(x) = A(x) \log(t) + \dots$$

$$\sum_{I=1}^{3} q_I(x) = \sum_{I=1}^{3} q_I(x)^2 + 8\,\pi\, A(x)^2 = 1$$

Fournodavlos-Rodnianski-Speck ‘23

Generalised Kasner spacetimes $(\overline{M} = (0, \infty) \times \mathbb{T}^3, h_{\mathsf{Kas}}, \phi_{\mathsf{Kas}})$ with positive Kasner exponents exhibit stable Big Bang formation within the Einstein scalar-field system.

Connecting both stability mechanisms

Any polarized $U(1)$-symmetric spacetime metric on $M = \overline{M} \times \mathbb{S}^1$ can be written as

$$h = e^{-2\sqrt{4\pi}\phi}\bar{g} + e^{2\sqrt{4\pi}\phi}(dx^3)^2$$

for a spacetime $(\overline{M} \cong I \times \Sigma, \bar{g})$ and $\phi : \overline{M} \to \mathbb{R}$.

Connecting both stability mechanisms

Any polarized $U(1)$-symmetric spacetime metric on $M = \overline{M} \times \mathbb{S}^1$ can be written as

$$h = e^{-2\sqrt{4\pi}\phi}\bar{g} + e^{2\sqrt{4\pi}\phi}(dx^3)^2$$

for a spacetime $(\overline{M} \cong I \times \Sigma, \bar{g})$ and $\phi : \overline{M} \to \mathbb{R}$.

Moncrief '86

(M, h) is a polarized $U(1)$-symmetric solution to the Einstein vacuum equations if and only if $(\overline{M}, \bar{g}, \phi)$ solves the Einstein scalar-field equations.

The reference solutions

Let (Σ, γ) be a closed (orientable) surface and $A < 0$.

The $2+1$ scalar field (FLRW) reference

$$\overline{M} = (0, t_0] \times \Sigma, \ \bar{g}_{FLRW} = -dt^2 + a(t)^2\gamma, \ \phi_{FLRW} = A \log(t)$$

a satisfies the Friedman equation

$$\dot{a}^2 = 4\,\pi\,A^2\,a^{-2} - \kappa$$

The reference solutions

Let (Σ, γ) be a closed (orientable) surface and $A < 0$.

The $2+1$ scalar field (FLRW) reference

$$\overline{M} = (0, t_0] \times \Sigma,\ \bar{g}_{FLRW} = -dt^2 + a(t)^2\gamma,\ \phi_{FLRW} = A\,\log(t)$$

a satisfies the Friedman equation

$$\dot{a}^2 = 4\,\pi\,A^2\,a^{-2} - \kappa$$

The $3+1$ vacuum solutions

$$M = (0, t_0] \times \Sigma \times \mathbb{S}^1,\ h = -dT^2 + b(T)^2\,\gamma + b_3(T)^2\,(dx^3)^2$$

Here, $b(T) \simeq T^{\frac{2}{3}},\ b_3(T) \simeq T^{-\frac{1}{3}}$ as $T \to 0$, with proportionality for $\Sigma \cong \mathbb{T}^2$.

Results

Past stability of FLRW solutions $2+1$ Einstein scalar-field system

Consider initial data on Σ_{t_0} to the $2+1$ Einstein scalar-field system close to FLRW data. Then, its past maximal globally hyperbolic development within the Einstein scalar-field equations admits a time function t such that the foliation $(\Sigma_s = t^{-1}(s))_{s\in(0,t_0]}$ is constant curvature. The Kretschmann scalar exhibits blow up of order t^{-4} as $t \to 0$.

Results

Past stability of FLRW solutions $2+1$ Einstein scalar-field system

Consider initial data on Σ_{t_0} to the $2+1$ Einstein scalar-field system close to FLRW data. Then, its past maximal globally hyperbolic development within the Einstein scalar-field equations admits a time function t such that the foliation $(\Sigma_s = t^{-1}(s))_{s\in(0,t_0]}$ is constant curvature. The Kretschmann scalar exhibits blow up of order t^{-4} as $t \to 0$.

- Result proven for Einstein scalar-field **Vlasov** system
- $\text{Riem}[g]$ is pure trace $\Rightarrow$ Evolution along CMC surfaces heavily simplifies

Results

Quiescent Big Bang for (some) polarized $U(1)$-symmetric spacetimes

Consider polarized $U(1)$-symmetric initial data on $\Sigma \times \mathbb{S}^1$ that is close to data for

$$h = e^{-2\sqrt{4\pi}\,\phi_{FLRW}}\,\bar{g}_{FLRW} + e^{2\sqrt{4\pi}\phi_{FLRW}}(dx^3)^2$$

for the $2+1$ FLRW solution $(\overline{M}, \bar{g}_{FLRW}, \phi_{FLRW} = A\,\log(t))$ with $A < 0$. Then, its past maximal globally hyperbolic development within the Einstein vacuum equations is polarized $U(1)$-symmetric and past C^2-inextendible. The Kretschmann scalar $e^{-4\sqrt{4\pi}\,\phi}\mathcal{K}[h]$ exhibits stable blow-up.

Results

Quiescent Big Bang for (some) polarized $U(1)$-symmetric spacetimes

Consider polarized $U(1)$-symmetric initial data on $\Sigma \times \mathbb{S}^1$ that is close to data for

$$h = e^{-2\sqrt{4\pi}\,\phi_{FLRW}}\,\bar{g}_{FLRW} + e^{2\sqrt{4\pi}\phi_{FLRW}}\,(dx^3)^2$$

for the $2+1$ FLRW solution $(\overline{M}, \bar{g}_{FLRW}, \phi_{FLRW} = A\,\log(t))$ with $A < 0$. Then, its past maximal globally hyperbolic development within the Einstein vacuum equations is polarized $U(1)$-symmetric and past C^2-inextendible. The Kretschmann scalar $e^{-4\sqrt{4\pi}\,\phi}\mathcal{K}[h]$ exhibits stable blow-up.

- Similar convergence properties for renormalized spacetime quantities
- In Kasner time T: $\mathcal{K}[h] \simeq T^{-4\pm c\,\varepsilon}$, ...
- Foliation is **not** CMC.

Thanks for listening!