Interacting Kerr-Newman electromagnetic fields

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Maxwell equations

- Complex electromagnetic fields:

$$\mathcal{F} = \mathbf{E} + i\mathbf{B} \tag{1}$$

- Maxwell equations have a form:

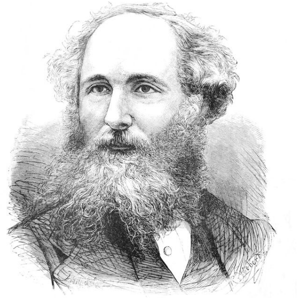
$$\nabla \cdot \mathcal{F} = \rho\,, \tag{2a}$$

$$i\partial_t \mathcal{F} - \nabla \times \mathcal{F} = 0 \tag{2b}$$

- For static fields, we use Ernst potential

$$\mathcal{Z} = \phi + i\psi, \tag{3}$$

$$\mathcal{F} = \mathrm{grad}\, \mathcal{Z} \tag{4}$$



James Clerk Maxwell (1831-1879)

Magic field–distorted Culomb solution

- Consider an ordinary Culomb solution in polar (ρ, φ, z) coordinates

$$f(R) = \frac{q}{\sqrt{\rho^2 + z^2}}$$

[1] E. T. Newman 1973 *Maxwell's equations and complex Minkowski space* J. Math. Phys. 14 102

[2] H. Erbin, *Janis-Newman Algorithm: Generating Rotating and NUT Charged Black Holes* Universe 2017, **3**, 19

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- Perform an imaginary shift

$$\widetilde{f}(r) = \frac{q}{\sqrt{\rho^2 + (z - ia)^2}}$$

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- Example of simply but highly non-trivial solution of Maxwell equations — limit of Kerr–Newman E-M field[1] for $M \to 0$
- Analog of Janis-Newman[2] algorithm for scalar field.

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Two points of view:

1. - Magic field as a discontinuous field on Euclidean space.
   - Electromagnetic field discontinuity $\rightarrow$ surface charge distribution on $\mathcal{D}$.

$$\mathcal{D} = \{\chi \in [0, a], z = 0\} \tag{5}$$

[3] D. Lynden-Bell. *A magic electromagnetic field* Chapter 25 in Stellar Astrophysical Fluid Dynamics p. 369-376. Cambridge Univ. Press, 2003
D. Lynden-Bell *Electromagnetic magic: The relativistically rotating disk* Phys. Rev. D, 70(10):105017, 2004. 20

[4] G. Kaiser *Distributional sources for Newman's holomorphic Coulomb field* J. Phys. A: Math. Theor., 37(36):8735, 2004.

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   - Detailed studies by Lynden–Bell (Magic field name coined)[3]
   - (Functional) analysis on singularities on $\mathcal{D}$ has been done by Kaiser[4]

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2. - Magic Field as a continuous field on extended Euclidean space.
   - Analytic continuation of Euclidean space without ring to double covered Riemann surface.

[5] Carter, B *Killing tensor quantum numbers and conserved currents in curved space* Phys. Rev. D, 16:3395-3414, Dec 1977.

[6] S. Tahvildar Zadeh S. *On a zero-gravity limit of the Kerr-Newman spacetimes and their electromagnetic fields* Journal of Mathematical Physics, 56(4):042501, 2015.

Two points of view:

2.
- Magic Field as a continuous field on extended Euclidean space.
- Analytic continuation of Euclidean space without ring to double covered Riemann surface.
- Related to the maximal analytic extension of Kerr-spacetime [5]
- Previously studied by Tahvildar Zadeh[6].

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Maximal analytic continuation

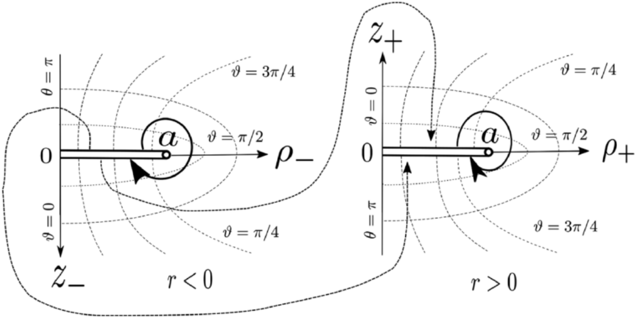

FIG. 1. Two charts (ρ_+, z_+) and (ρ_-, z_-) needed to cover the geometry (2.4) with $r \in (-\infty, \infty)$. Each chart has a branch cut along the $[0, a]$ segment of the ρ-axis. Lines of constant r are oblate (half-)ellipses, orthogonal to them are hyperbolas of constant ϑ. The r, ϑ coordinates are discontinuous through the cut on each individual chart, but they smoothly continue from one chart to the other if the upper edge of one cut is glued to the lower edge of the other and vice versa as shown. A contour around the branch point $(a, 0)$ then performs one revolution in the (ρ_+, z_+) chart, followed by a second revolution in the (ρ_-, z_-) chart, and only after that closes.

Gibbons, Volkov, *Zero mass limit of Kerr spacetime is a wormhole* PRD (2017).

Lagrangian for static fields

(with S. Aghapour, L. Andersson and K. Rosquist)

$$\mathcal{L} = \frac{1}{2\Omega}\int_{\mathcal{V}} \mathcal{F}^2 \,\mathrm{d}V = \frac{1}{2\Omega}\int_{\mathcal{V}} \boldsymbol{\nabla}\mathcal{Z}\cdot\boldsymbol{\nabla}\mathcal{Z}\mathrm{d}V = \frac{1}{4\Omega}\oint_{\partial\mathcal{V}} \boldsymbol{\nabla}\mathcal{Z}^2\cdot\mathrm{d}\boldsymbol{S},$$

where $\mathcal{F} = E + \imath B = \operatorname{grad}\mathcal{Z}$.

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- The Lagrangian of the magic field (double covered surface)

$$\mathcal{L} = -\frac{\pi q^2}{16a}.$$

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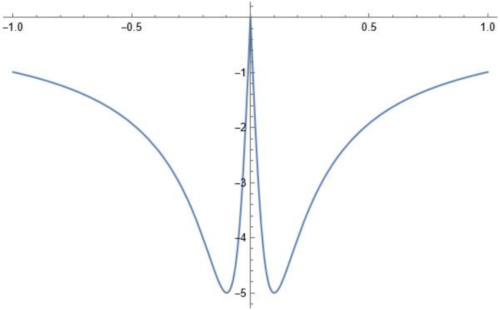
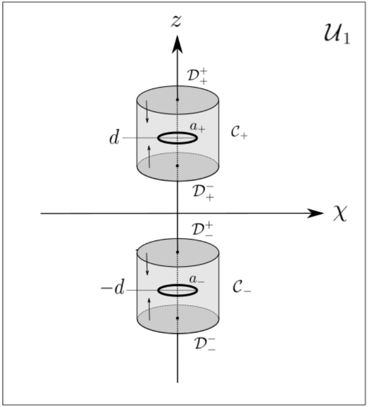
- For interacting Fields, $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, Lagrangian $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{int}$ where

$$\mathcal{L}_{\text{int}} = \frac{1}{\Omega}\int_{\mathcal{V}} \mathcal{F}_1\cdot\mathcal{F}_2 \, \mathrm{d}V = \frac{1}{2\Omega}\oint_{\partial\mathcal{V}} \boldsymbol{\nabla}\left(\mathcal{Z}_1\mathcal{Z}_2\right)\cdot\mathrm{d}\boldsymbol{S}.$$

Up-down configuration

$$L_{int} = (q_+ q_-) \frac{2d}{4d^2 + (a_+ + a_-)^2}$$

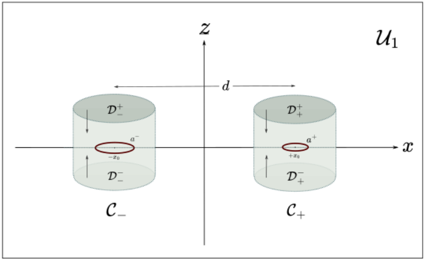




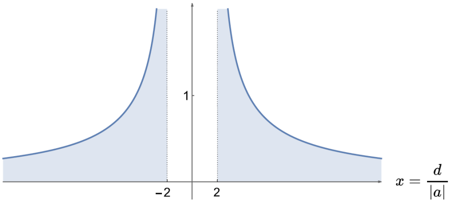
Side by side configuration

$$L_{int} = \left(\frac{q_+ q_-}{2df}\right) \frac{d(d+2f) - (a_+ - a_-)^2}{\sqrt{(d+2f)^2 - 4e^2}}$$

where $f = \frac{1}{2d}\sqrt{(d^2-(a_+-a_-)^2)(d^2-(a_++a_-)^2)}\,; \quad e = \frac{1}{2d}\left(a_+^2 - a_-^2\right)\,.$



For $a_\pm = a\,, q_1 q_2 > 0\,, L = \frac{q_1 q_2}{\sqrt{d^2 - 4a^2}} > 0$



References

- One sheet approach with source
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- Two-sheeted extension of "Euclidean" space
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  - Kiessling, Tahvildar-Zadeh, *Dirac's Point Electron in the zero-Gravity Kerr-Newman* World, (2015).

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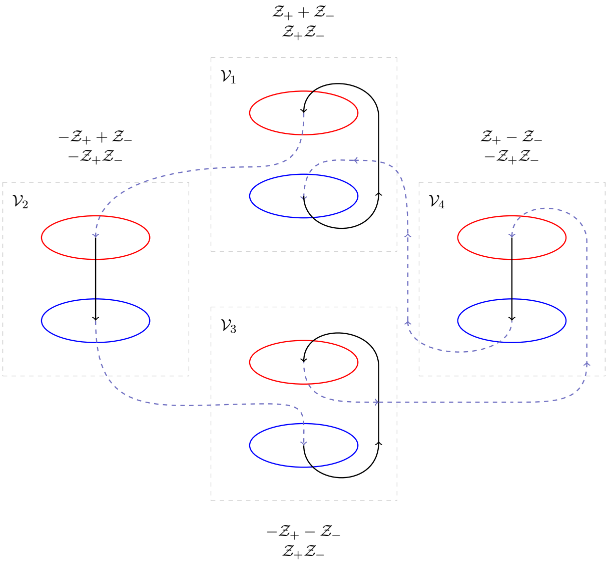
Thank You!

Topology of interacting fields



Oblate spheroidal coordinates

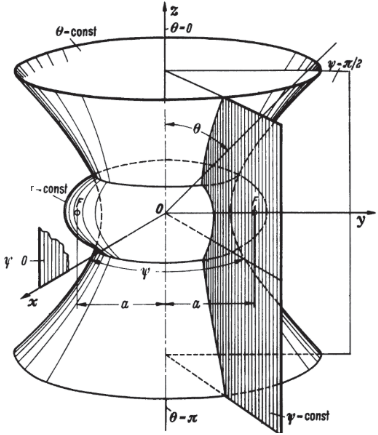
$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi,$$
$$y = \sqrt{r^2 + a^2}\sin\theta\sin\phi, \qquad (6)$$
$$z = r\cos\theta$$

$$\frac{x^2}{r^2 + a^2} + \frac{y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1$$

(oblate spheroids, $r = \text{const}$),

$$\frac{x^2}{a^2\sin^2\theta} + \frac{y^2}{a^2\sin^2\theta} - \frac{z^2}{a^2\cos^2\theta} = 1$$

(hyperboloids of one sheet, $\theta = \text{const}$),



Oblate spheroidal coordinates (η, θ, φ).
Oblate spheroids ($\eta = \text{const}$),
hyperboloids of revolution ($\theta = \text{const}$),
half-planes ($\varphi = \text{const}$).

Invariants of magic field

Energy density of electromagnetic field is

$$\mathcal{E} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) = \frac{e^2\left(r^2 + a^2 + a^2\sin^2\vartheta\right)}{2\Sigma^3}$$

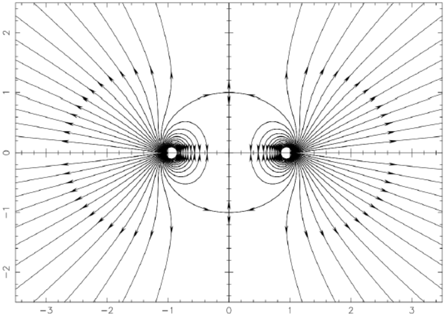
The total energy $\int_\Omega \mathcal{E}$ diverges:

$$\lim_{\epsilon\to 0}\int_\epsilon^\infty dr\int_0^\pi d\vartheta\sqrt{-g}\mathcal{E} = \lim_{\epsilon\to 0}\int_\epsilon^\infty dr\frac{e^2\left(a\arctan\left(\frac{a}{r}\right) + r\right)}{r^3} \to \infty$$

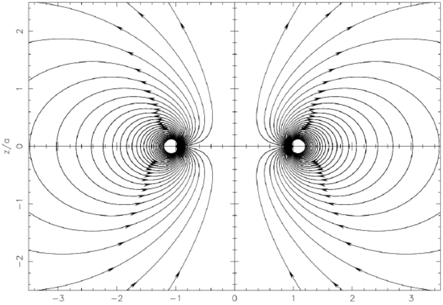
Magic field

The explicit form of electromagnetic fields:

$$E = \frac{q}{\sum^2}\left(\left(r^2 - a^2\cos^2\theta\right)dr - a^2 r\sin 2\theta d\theta\right)$$
$$B = \frac{-q}{\Sigma^2}\left(2ar\cos\theta dr + a\left(r^2 - a^2\cos^2\theta\right)\sin\theta d\theta\right) \tag{7}$$



Electric field lines



Magnetic field lines

Lower-dimensional toy model

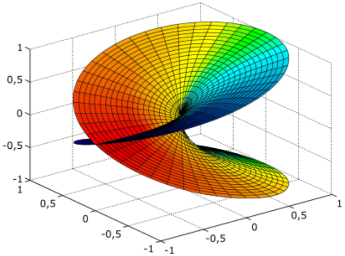
- In polar representation, $z = \chi e^{\imath\varphi}$, the one over square-root function:

$$f : (\chi, \varphi) \to \frac{1}{\sqrt{\chi}} e^{-\imath\varphi/2} \tag{8}$$

- For a complex plane $(\chi \in \mathbb{R}_+, \varphi \in]-\pi, \pi])$, the function is not defined at the origin, and not continuous on the cut.
- Considering

$$\begin{aligned} \widetilde{f} &: \mathbb{R}_+ \times]-2\pi, 2\pi[\to \mathbb{C}\,, \\ \widetilde{f} &: (\chi, \widetilde{\varphi}) \to \frac{1}{\sqrt{\chi}} e^{\imath\widetilde{\varphi}/2}\,, \end{aligned} \tag{9}$$

is continuous everywhere.



Riemann surface for $f(z) = 1/\sqrt{z}$.

The Janis-Newman[7] (J-N) algorithm

The Reissner–Nordström spacetime:

- Electrovacuum solution of the Einstein equations
- Static, charged black hole

Physical characteristics – parameters:

- M mass
- Q charge

The Kerr–Newman spacetime:

- Electrovacuum solution of the Einstein equations
- Rotating, charged black hole

Physical characteristics – parameters:

- M mass
- Q charge
- $a \quad (= J/M)$ angular momentum per unit mass
- $\mu \quad (= aQ)$ magnetic moment

[7] H. Erbin, *Janis-Newman Algorithm: Generating Rotating and NUT Charged Black Holes* Universe 2017, **3**, 19