Approximately conserved quantities in GR

Berend Schneider

Conserved quantities associated with the wave equation

$$\partial_u \partial_v (r\phi) = -l(l+1)\frac{r\phi}{r^2}$$

$$\partial_u \left(r^{-2l} \partial_v (r^2 \partial_v)^l (r\phi) \right) = 0$$

You will learn:

Part 1: Why should you care?

- The application: asymptotics of the wave equation

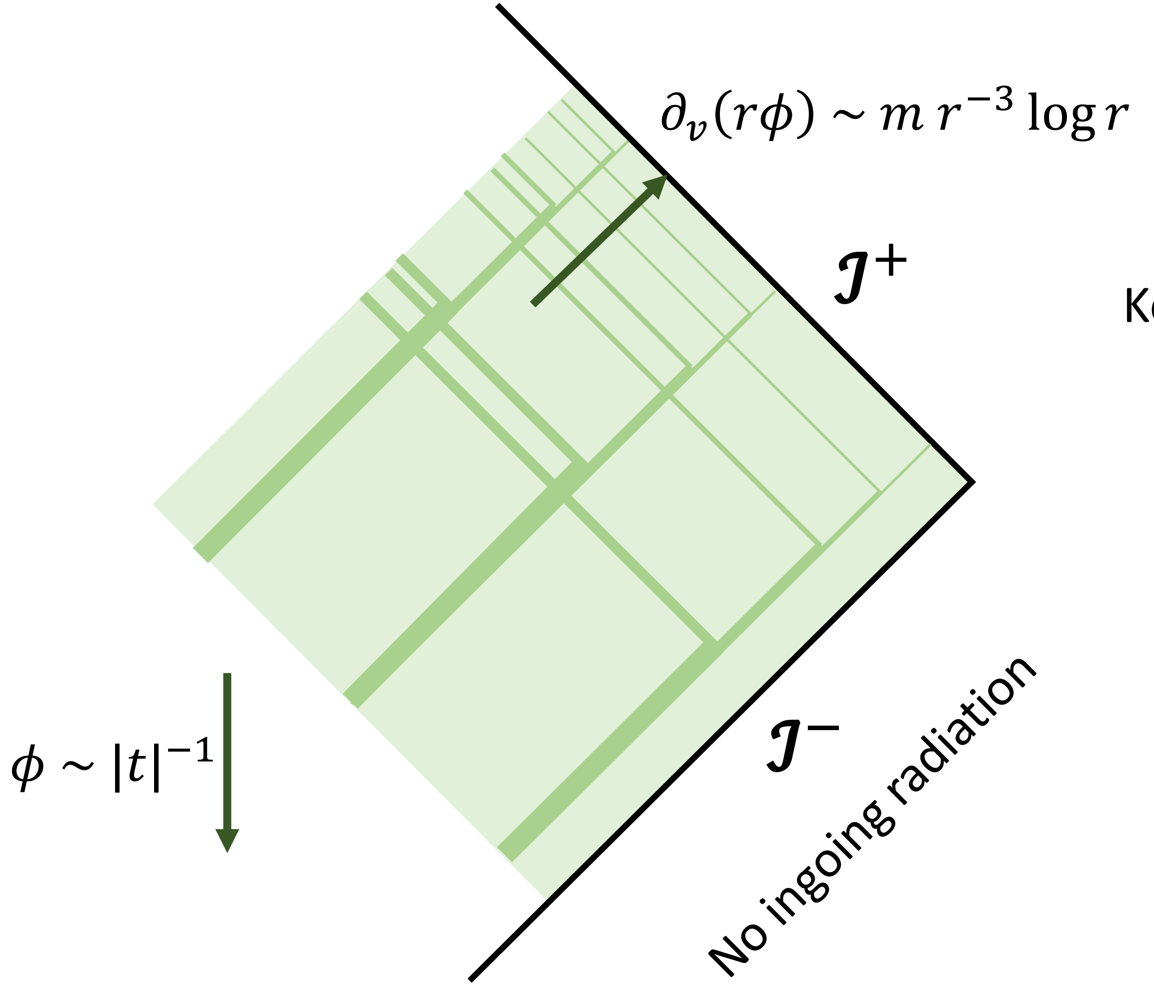
Part 2: *What* are NP constants?

- A slightly more formal derivation

Part 3: How to generalize NP constants to non-flat spaces

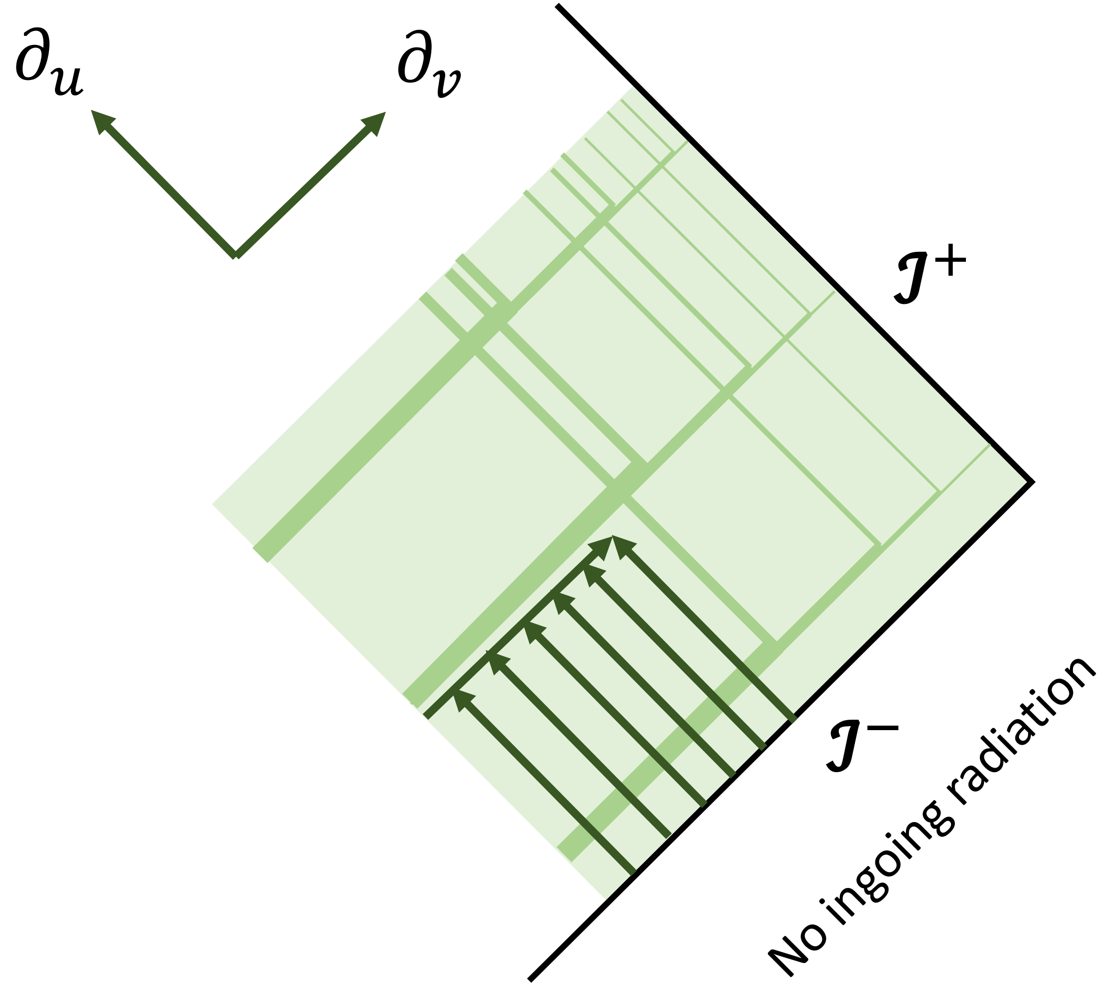
- Conformal trickery

Why you should care



Kehrberger, L. *The Case Against Smooth Null Infinity I*

Why you should care



$$\partial_u \partial_v (r\phi) = O(r^{-3}) r\phi$$

NP-like charges in Kerr

Late-time tails and mode coupling of linear waves on Kerr spacetimes

Yannis Angelopoulos *1, Stefanos Aretakis†2, and Dejan Gajic ‡3,4

$$\frac{(r^2+a^2)^2}{\Delta}P_0 := \frac{(r^2+a^2)^2}{\Delta}L\phi_0 - \frac{1}{4}a^2\pi_0(\sin^2\theta T\phi),$$

$$\frac{(r^2+a^2)^2}{\Delta}P_1 := \frac{(r^2+a^2)^2}{\Delta}L\check{\phi}_1^{(1)} - \frac{1}{4}\left[a^2\pi_1(\sin^2\theta T\check{\phi}^{(1)}) + 2a\Phi\check{\phi}_1^{(1)}\right]$$
$$- \frac{1}{4}[-\alpha - \alpha_\Phi\Phi][a^2\pi_1(\sin^2\theta T\phi) + 2a\Phi\phi_1],$$

$$\frac{(r^2+a^2)^2}{\Delta}P_2 := \frac{(r^2+a^2)^2}{\Delta}L\check{\phi}_2^{(2)} - \frac{1}{4}\left[a^2\pi_2(\sin^2\theta T\check{\phi}^{(2)}) + 2a\Phi\check{\phi}_2^{(2)}\right]$$
$$- \frac{1}{4}[M-\gamma-a\Phi][a^2\pi_2(\sin^2\theta T\check{\phi}^{(1)}) + 2a\Phi\check{\phi}_2^{(1)}]$$
$$- \frac{1}{4}[2\beta + 2\beta_{\Phi^2}\Phi^2 + \gamma(a\Phi - M)][a^2\pi_2(\sin^2\theta T\phi) + 2a\Phi\phi_2].$$

Corollary 4.2. *Let $\psi \in C^\infty(\mathcal{R} \to \mathbb{C})$ denote a solution to (3.1) that is supported on a fixed azimuthal mode with azimuthal number m, i.e. it satisfies $\Phi\psi = im\psi$. Then:*

$$4\underline{L}P_0 = O_\infty(r^{-3})\phi_0 + a^2O_\infty(r^{-3})T\pi_0(\sin^2\theta\phi) + a^2O_\infty(r^{-2})LT\pi_0(\sin^2\theta\phi), \tag{4.10}$$

$$4\underline{L}P_1 = [-4r^{-1} + O_\infty(r^{-2})]P_1 + O_\infty(r^{-3})[\check{\phi}_1^{(1)} + \phi_1] + a^2O_\infty(r^{-3})[T\pi_1(\sin^2\theta\check{\phi}^{(1)}) + T\pi_1(\sin^2\theta\phi)] \tag{4.11}$$
$$+ a^2O_\infty(r^{-2})[LT\pi_1(\sin^2\theta\check{\phi}^{(1)}) + LT\pi_1(\sin^2\theta\phi)],$$

$$4\underline{L}P_2 = [-8r^{-1} + O_\infty(r^{-2})]P_2 + O_\infty(r^{-3})[\check{\phi}_2^{(2)} + \check{\phi}_2^{(1)} + \phi_2] \tag{4.12}$$
$$+ a^2O_\infty(r^{-3})[T\pi_2(\sin^2\theta\check{\phi}^{(2)}) + T\pi_2(\sin^2\theta\check{\phi}^{(1)}) + T\pi_2(\sin^2\theta\phi)]$$
$$+ a^2O_\infty(r^{-2})[LT\pi_2(\sin^2\theta\check{\phi}^{(2)}) + LT\pi_2(\sin^2\theta\check{\phi}^{(1)}) + LT\pi_2(\sin^2\theta\phi)],$$

where we allow the constants in the terms in $O_\infty(r^{-k})$, $k = 0, 1$, to depend also on m.

What are NP constants?

$$\Box\,\phi_k = (þ'_c þ_c - ð'ð)\phi_k$$

$$\Box\,\phi_k^{l=s} = 0 \;\text{ and }\; ð'ð\phi_k^{l=s} = 0 \Longrightarrow þ'_c þ_c \phi_k^{l=s} = 0$$

What are NP constants? II: Higher modes

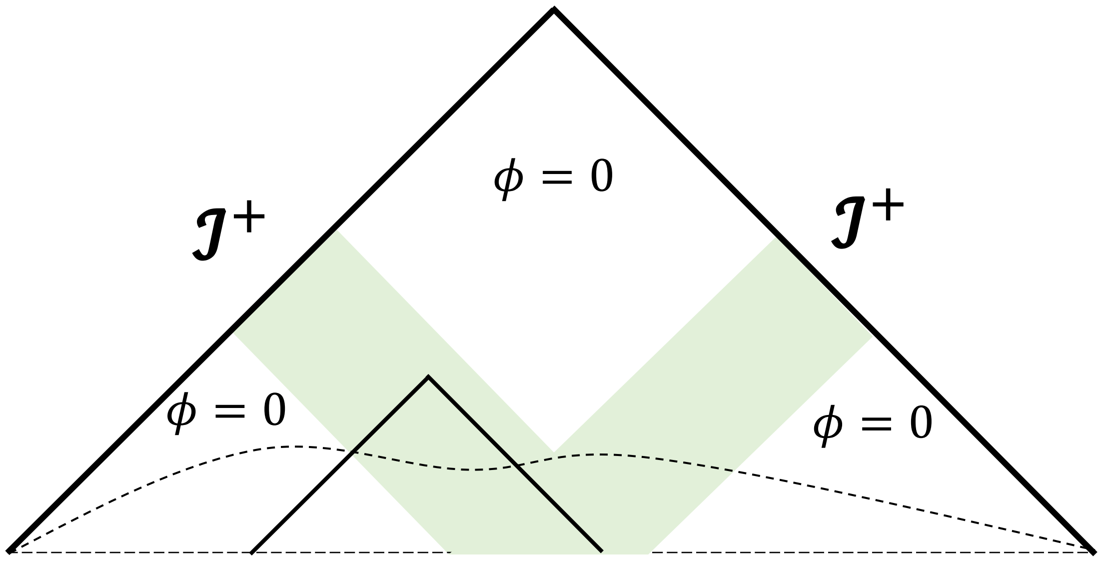
$$[\Box, ð'^{\dagger} þ_c] = 0 \text{ and } ð'^{\dagger} {}_sY_l \propto {}_{s+1}Y_l$$

$$\Longrightarrow þ'_c þ_c ð'^{\dagger} þ_c \phi_k^{l=s+1} = 0$$

$$\Longrightarrow þ'_c þ_c \left(ð'^{\dagger} þ_c\right)^{l-s} \phi_k^{l} = 0$$

What are NP constants? III: A picture

$$þ'_c þ_c \left(ð'^{\dagger}_c þ_c\right)^{l-s} \phi^l_k = 0$$



Generalized NP charges

Non-conformal quantities

$\rho = O(r^{-1})$

$\tau = O(r^{-2})$

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Conformal quantities

$\rho - \bar{\rho} = O(r^{-2})$

$\tau - \bar{\tau}' = O(r^{-3})$

$þ_c \coloneqq þ + (w - b)\rho - q(\rho - \bar{\rho})$

$ð_c \coloneqq ð + (w - s)\tau + q(\tau - \bar{\tau}')$

Generalized NP charges

$$\square := (\text{þ}' - \bar{\rho}')(\text{þ} - (1+2s)\rho) - (\text{ð}' - \bar{\tau})(\text{ð} - (1+2s)\tau)$$
$$= \text{þ}'_c \text{þ}_c - \text{ð}'_c \text{ð}_c$$

$$[\square, \text{ð}'^{\dagger}_c \text{þ}_c] = 6s\Psi_2 \text{ð}'^{\dagger}_c \text{þ}_c = O(r^{-3})$$

Generalized NP charges

$$Q_l = \text{þ}_c \left(\eth_c'^{\dagger} \text{þ}_c\right)^{l-s} \phi_k^l$$

$$\text{þ}_c' Q_l = \sum_{n=s+1}^{l} 6n \left(\eth_c'^{\dagger} \text{þ}_c\right)^{l-n} \left(\Psi_2 \left(\eth_c'^{\dagger} \text{þ}_c\right)^{n-s} \phi_k^l\right) = O(r^{-3-m}) \text{þ}_c^m \phi_k^l$$

Thank you for listening!

Questions?