The Initial Value Formulation of the Einstein Equations in Generalized Geometry

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Motivation

Generalized geometry incorporates the B-field and the Dilaton ϕ into the geometry.

The related equations of motion from supergravity can be stated as the generalized Einstein equations (Coimbra, Strickland-Constable, and Waldram 2011):

$$\mathcal{R}c = 0 \qquad \text{and} \qquad \mathcal{S}c = 0$$

Similarly, one would like to have a geometric description of initial data for these equations

-> Generalized Submanifold Geometry!

Eventually prove the existence of an MGHD in this context.

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The Generalized Tangent Bundle

On $TM \oplus T^*M$, we have the natural inner product of neutral signature

$$\langle X + \xi, X + \xi \rangle = \xi(X), \qquad X + \xi \in \Gamma(TM \oplus T^*M)$$

Additionally to diffeomorphism-invariance, $\langle \cdot, \cdot \rangle$ exhibits invariance under B-field transformations:

$$e^B(X + \xi) := X + \xi + B(X), \qquad B \in \Omega^2(M)$$

In generalized geometry, B-field invariance is as fundamental as diffeomorphism invariance!

Note that the anchor

$$\pi \colon T \oplus T^* \longrightarrow T, \qquad \pi(X + \xi) := X.$$

is invariant as well: $\pi \circ e^B = \pi$.

The H-twisted Bracket

Equipping $T \oplus T^*$ with a closed three-form $H \in \Omega^3(M)$, the H-twisted bracket provides an anologue of the Lie derivative:

$$[X + \xi, Y + \eta]_H = L_X(Y + \eta) - i_Y d\xi + H(X, Y)$$

H is needed to deal with B-field transformations, as

$$e^{-B}\left[e^B(a),\ e^B(b)\right]_H = [a, b]_{H+\mathrm{d}B}, \qquad a, b \in \Gamma(E)$$

The bracket satisfies, denoting $a, b, c \in \Gamma(T \oplus T^*)$

(i) a Jacobi identity,

(ii) $\pi c(\langle a, b\rangle) = \langle [c, a]_H, b\rangle + \langle a, [c, b]_H\rangle$,

(iii) $[a, b]_H + [b, a]_H = 2\mathrm{d}\langle a, b\rangle$.

Generalized Geometry

A generalized metric $\mathcal{G}$ is a non-degenerate symmetric bilinear form on $E \equiv T \oplus T^*$, such that

$$\mathcal{G}^{-1}(\langle a, \cdot \rangle, \langle b, \cdot \rangle) = \mathcal{G}(a, b), \qquad a, b \in \Gamma(E)$$

It is equivalent to a tuple (e^B, g).

A generalized connection is a map $D\colon \Gamma(E) \to \Gamma(E^* \otimes E)$ s.t.

(i) $D_a(fb) = (\pi a)(f)b + f\, D_a b$,

(ii) $\pi a(\langle b, c \rangle) = \langle D_a b, c \rangle + \langle b, D_a c \rangle$.

It is generalized Levi-Civita with respect to a Dilaton $e \in \Gamma(E)$ if

$$D\mathcal{G} = 0, \qquad T[D] = 0, \qquad \mathrm{tr}\,(Da) = \mathrm{div}^g(\pi a) + \langle e, a \rangle$$

where $a \in \Gamma(E)$ arbitrary.

Generalized LC always exists, but not uniquely!

Generalized Curvature

Given a generalized connection D, one can define

$$\mathcal{R}m^D(a,b,v,w) \coloneqq \frac{1}{2}\Big\{ \langle D_v D_w b - D_w D_v b - D_{D_v w} b + D_{D_w v} b, a\rangle + \ldots \Big\}$$

$$\overline{\mathcal{R}c}^D(a,b) \coloneqq \mathrm{tr}_{\langle\cdot,\cdot\rangle}\, \mathcal{R}m^D(\cdot, a, \cdot, b)$$

$$\mathcal{S}c \coloneqq \mathrm{tr}_{\mathcal{G}}\, \overline{\mathcal{R}c}$$

The generalized Einstein equations

$$\mathcal{R}c \coloneqq \overline{\mathcal{R}c}^D\Big|_{E_\pm \oplus E_\mp} = 0 \qquad \text{and} \qquad \mathcal{S}c = 0$$

are equivalent to, assuming $e = 2\mathrm{d}\phi$ where $\phi \in C^\infty(M)$,

$$\mathrm{Rc} = \frac{1}{4}H^2, \qquad \mathrm{d}^*H = -H(\mathrm{grad}_g\, \phi) - \nabla\nabla\phi, \qquad \frac{|H|_g^2}{6} = \mathrm{d}^*\mathrm{d}\phi + |\mathrm{d}\phi|_g^2$$

Generalized Submanifold Geometry

Let $E := TM \oplus T^*M$ and $\Sigma \hookrightarrow M$ a hypersurface.

Then one can consider the generalized tangent bundle $E_\Sigma := T\Sigma \oplus T^*\Sigma$ (Bursztyn, Cavalcanti, and Gualtieri 2007).

Theorem (Cortés, S-, 2025)

The following geometrical objects on E reduce in a natural way to corresponding objects on E_Σ:

$$\mathcal{G} \longrightarrow \mathcal{H}, \qquad \mathrm{div} \longrightarrow \mathrm{div}_\Sigma, \qquad D \longrightarrow D^\Sigma$$

There exist LC connections on E that reduce to LC connections on E_Σ.

Generalized Exterior Curvature

Given the unit normal n on Σ, the vectors $n_\pm = n \pm n^\flat \in \Gamma(E)$ are such that

$$E = E_\Sigma \oplus \mathrm{span}\{n_+, n_-\}$$

is an orthogonal decomposition.

We define the generalized second fundamental form as

$$\overline{\mathcal{K}}^{n_\pm}(a,b) \coloneqq \mathcal{G}(D_a n_\pm, b), \qquad \mathcal{T} \coloneqq \mathrm{tr}_{\mathcal{H}}\, \overline{\mathcal{K}}^{n_\pm}$$

Theorem (Cortés, S-, 2025)

The constraint equations for the generalized Einstein equations are

$$\mathcal{R}c^\pm(a_\mp, n_\pm) = (\mathrm{div}_\Sigma \hat{\mathcal{K}}^\pm)(a_\mp) - \pi a_\mp(\mathcal{T})$$

$$2\mathcal{R}c^\pm(n_\mp, n_\pm) + \mathcal{S}c = \mathcal{S}c_\Sigma - \left|\mathcal{K}^\pm\right|^2 + \mathcal{T}^2$$

Literature

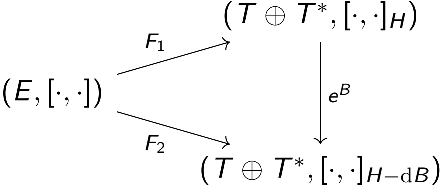
Coimbra, Andre, Charles Strickland-Constable, and Daniel Waldram (2011). "Supergravity as generalised geometry I: type II theories". In: *Journal of High Energy Physics* 2011.11, pp. 1–35.

Bursztyn, Henrique, Gil R. Cavalcanti, and Marco Gualtieri (2007). "Reduction of Courant algebroids and generalized complex structures". In: *Advances in Mathematics* 211.2, pp. 726–765.

Exact Courant Algebroids

An exact Courant algebroid is a tuple $(E \to M, \pi\colon E \to T, \langle\cdot,\cdot\rangle, [\cdot,\cdot])$ such that one can find a VB isomorphism $F\colon E \to T \oplus T^*$ relating the structure on E to the corresponding structure on $T \oplus T^*$ for some $H \in \Omega^3_{\mathrm{cl}}(M)$.

Crucially, there is no canonical choice of F, and any two choices differ by a B-field transformation!



B-field invariance is built into the exact CA!