Insights on Benoit Michel Charges on Asymptotically Hyperboloidal Initial Data Sets and their evolution

Anna Sancassani, joint with
Saradha Senthil-Velu

Eberhard Karls Universität Tübingen

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- $3+1$-decomposition of spacetime
- Definition of energy and linear momentum a lá Michel on asymptotically hyperboloidal initial data
- Hyperboloidal evolution
- Evolution of energy and linear momentum
- Comparison to previous results

3+1 - DESCRIPTION OF SPACETIME

Let $(\mathcal{M}, \gamma)$ be a vacuum spacetime with time-function $t : \mathcal{M} \to (a, b) \subseteq \mathbb{R}$. Then, $\boldsymbol{\Sigma} := \{\Sigma_t\}_{t\in(a,b)}$ is a foliation of $\mathcal{M}$ by spacelike hypersurfaces.

Each leaf Σ of $\boldsymbol{\Sigma}$, with induced metric ${}^{\Sigma}g$ and second fundamental form ${}^{\Sigma}K$ is a vacuum **initial data set** for the Einstein Equation, i.e.

$$\Phi(g, K) := \begin{pmatrix} \mathrm{R} + (\mathrm{tr}_g K)^2 - |K|_g^2 \\ 2\mathrm{div}_g(K - \mathrm{tr}_g K \cdot g) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

Φ is called **constraint operator**.

We define coordinates (t, x^i) on $V := (t_0 - \varepsilon, t_0 + \varepsilon) \times U \subseteq M$ for some $\varepsilon > 0$ by flowing coordinates (x^i) defined on an open set $U \subseteq \Sigma_{t_0}$ along any timelike vector field $\xi \in \Gamma(TM)$ with $dt(\xi) = 1$.

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The vector field $\partial_t = \xi$ splits uniquely as $\partial_t =: N\nu + \mathbf{X}$, with

- $N : U \to \mathbb{R}_+$ smooth **lapse (function)**, and
- $\mathbf{X} \in \Gamma(TV)$ smooth, spacelike **shift (vector field)** $X_p \parallel \Sigma_{t(p)}\ \forall p \in V$.

Then, on $(t_0 - \varepsilon, t_0 + \varepsilon) \times U$, the spacetime metric γ can be written as

$$\gamma = -\hat{N}^2 dt^2 + {}^{\Sigma}\mathbf{g}_{ij}\,(dx^i + X^i dt)(dx^j + X^j dt) \tag{2}$$

and the evolution of induced metric and 2nd fundamental form along the foliation is described by the *Einstein Evolution Equations*

$$\begin{cases} \mathcal{L}_\nu {}^{\Sigma}\mathbf{g} = 2\,{}^{\Sigma}\mathbf{K} \\ \mathcal{L}_\nu {}^{\Sigma}\mathbf{K} - \frac{{}^{\Sigma}\boldsymbol{\nabla}^2 N}{N} = 2({}^{\Sigma}\mathbf{K})^2 - {}^{\Sigma}\mathbf{Ric} - (\mathrm{tr}\,{}^{\Sigma}\mathbf{K})\,{}^{\Sigma}\mathbf{K} \end{cases} \tag{3}$$

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Energy and linear momentum on asymptotically hyperboloidal Initial data

Definition

A vacuum initial data set (Σ, g, K) is **asymptotically hyperboloidal** if there exist $\mathcal{K} \subset \Sigma, \mathcal{K}_0 \subset \mathbb{H}^3$ compact, and a diffeomorphism at infinity

$$\Psi : \mathbb{H}^3 \setminus \mathcal{K}_0 \to \Sigma \setminus \mathcal{K},$$

such that the $(0, 2)$-symmetric tensors on $\mathbb{H}^3 \setminus K_0$

$$\dot{g} := \Psi^* g - b \text{ and } p := \Psi^* K - \Psi^* g$$

have the following asymptotic behavior as $r \to \infty$

$$\dot{g}_{rr} = \frac{\mathbf{m}_{rr}}{r^5} + O(r^{-6}), \qquad \dot{g}_{\alpha r} = O(r^{-3}), \qquad \dot{g}_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \frac{{}^g\mathbf{m}_{\alpha\beta}}{r} + O(r^{-2}),$$

$$p_{rr} = O(r^{-5}), \qquad p_{\alpha r} = O(r^{-3}), \qquad p_{\alpha\beta} = p^{(0)}_{\alpha\beta} + \frac{{}^k\mathbf{m}_{\alpha\beta}}{r} + O_2(r^{-2}).$$

Refer [Chen, Wang **and** Yau 2014].

Given (Σ, g, K), (Σ_0, g_0, K_0) vacuum initial data

- $\Psi : \Sigma_0 \setminus \mathcal{K}_0 \to \Sigma \setminus \mathcal{K}$
- $\Phi : \Gamma(\mathcal{M} \times S_2\Sigma) \to \Gamma(\mathbb{R} \oplus T^*\Sigma)$ constraint operator

Then, given a test function $\mathcal{V}$ and small $e := \Psi^*(g, K) - (g_0, K_0)$

$$\langle \mathcal{V}, \Phi(\Psi^*(g,K)) - \Phi_0 \rangle_0 = \langle \mathcal{V}, D\Phi_0(e) \rangle_0 + \underbrace{\langle \mathcal{V}, Q(e) \rangle_0}_{=:Q(\mathcal{V},e)}$$

$$= \mathrm{div}_0 \mathbb{U}(\mathcal{V}, e) + \langle D\Phi_0^*(\mathcal{V}), e \rangle_0 + Q(\mathcal{V}, e).$$

Definition

Let Ψ be given and let $\mathcal{V} \in \ker(D^*\Phi_0)$. We say that there exists a well-defined total charge on (Σ, g, K) associated to $\mathcal{V}$ w.r.t. Ψ if $\langle \mathcal{V}, \Phi(\Psi^*(g,K)) - \Phi_0 \rangle_0$ and $Q(\mathcal{V}, e)$ are integrable w.r.t the volume density induced by g_0.

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Theorem and definition [Michel 2011]

Assume that there exists a well-defined total charge on (Σ, g, K) associated to $\mathcal{V} \in \ker(D^*\Phi_0)$ w.r.t. Ψ. Then,

$$m((g,K),\Psi,\mathcal{V}) := \lim_{k\to\infty} \int_{S_k} \mathbb{U}(\mathcal{V},e)(n)\,\mathrm{d}S. \tag{4}$$

$(B_k)_{k\in\mathbb{N}}$ is a non-decreasing exhaustion of $\mathbf{\Sigma_0}$ such that each B_k has smooth compact boundary S_k, n and $\mathrm{d}S$ are the outer normal and surface measure of S_k w.r.t. $\mathbf{g_0}$. The total charge is invariant under changes of diffeomorphisms at infinity that are asymptotic to the identity.

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$\ker(D^*\Phi_0) \quad \longleftrightarrow \quad$ KVFs

Refer [Moncrief, 1975], [Berger, 1976]

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$$\text{KIDs} := \ker(D^*\Phi_0) \longleftrightarrow \text{KVFs}$$

Refer [Moncrief, 1975], [Berger, 1976], [Beig **and** Chruściel 1997], [Maerten 2004]

Energy and Linear Momentum in AH IDS

The energy of an AH IDS is given by:

$$E = \frac{1}{32\pi} \int_{\mathbb{S}^2} \underbrace{\left(3\mathrm{tr}_\sigma {}^g\mathbf{m}_{\alpha\beta} + 2\mathrm{tr}_\sigma {}^k\mathbf{m}_{\alpha\beta} + 2\mathbf{m}_{rr}\right)}_{\text{mass aspect}} dA_{\mathbb{S}^2}$$

and the linear momentum of an AH IDS is given by:

$$P^i = \frac{1}{32\pi} \int_{\mathbb{S}^2} \left(3\mathrm{tr}_\sigma {}^g\mathbf{m}_{\alpha\beta} + 2\mathrm{tr}_\sigma {}^k\mathbf{m}_{\alpha\beta} + 2\mathbf{m}_{rr}\right) x^i dA_{\mathbb{S}^2},$$

where x^i are the first spherical harmonics on the unit sphere.

Refer [Chen, Wang **and** Yau 2014],

PREVIOUS RESULTS

- Hamiltonian analysis in fixed asymptotically Minkowskian coordinates in space-time. [Trautman 1958]
- Space-time Bondi coordinates [Bondi, van der Burg **and** Metzner-Sachs, 1962]
- Space-time "charge integrals", derived in a geometric Hamiltonian framework [Chruściel 1985], [Chruściel Jezierski **and** Kijowski 2001, Chruściel **and** Herzlich 2003,]
- Initial data charge integrals [Chruściel, Jezierski **and** Leski 2004]
- Optimal Isometric Embeddings - Liu Yau mass [Chen, Wang **and** Yau 2014]
- Stronger asymptotics - Wang's asymptotics e.g. [Sakovich 2021] and others

Evolution of charges

We define the hyperboloidal temporal functions in Minkowski $(\mathbb{R}^{3,1}, \eta)$:

$$\tau := t + h(r).$$

and require that for some $C > 0$

$$|h'(r)| = 1 - \frac{C}{r^2} + O_2(r^{-3}).$$

Then, the $\tau =$ const hypersurfaces are hyperboloids of the same radius.
[Zenginoğlu 2008], [Valiente Kroon, Gomes Da Silva 2024], ...

We study the evolution in direction $\partial_\tau = \frac{1}{N}(\nu - X)$, with

$$N = r + O_2(r^{-1}), \qquad X = -r^2\partial_r + O_1(r^0).$$

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Define $\mathcal{H} := \{(N, X) \in \mathcal{C}^\infty(\Sigma) \times \Gamma(T\Sigma)|$ hyperboloidal foliation $\}$

Theorem

Let (M, g, K) be an asymptotically hyperboloidal initial data set. Assume (M_τ, g_τ, K_τ) is the AH foliation of the globally hyperbolic development of (M, g, K) defined by $(N, X) \in \mathcal{H}$.

Then, the loss of energy and linear momentum along this foliation is

$$\frac{d}{d\tau}E = -\frac{1}{4\pi}\int_{\mathbb{S}^2} |{}^{(0)}g_{\alpha\beta} + {}^{(0)}p_{\alpha\beta}|^2 dA_{\mathbb{S}^2}, \tag{5}$$

and

$$\frac{d}{d\tau}P^i = -\frac{1}{4\pi}\int_{\mathbb{S}^2} |{}^{(0)}g_{\alpha\beta} + {}^{(0)}p_{\alpha\beta}|^2 x^i dA_{\mathbb{S}^2}. \tag{6}$$

[Zenginoğlu 2008]

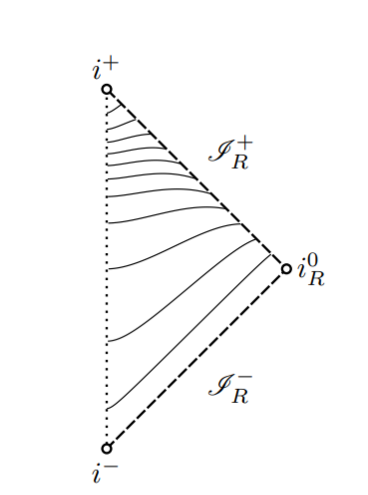

Figure: Hyperboloidal foliation

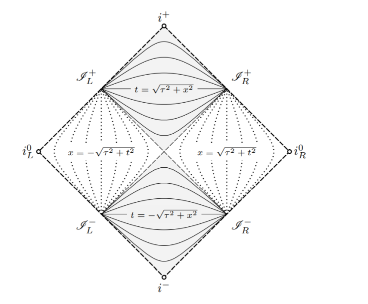

Figure: Milne foliation

Comparison to the Null Setting

Consider a spacetime $(\mathcal{M}, \gamma)$. In Bondi-Sachs coordinates, $\mathcal{I}^+ = I \times \mathbb{S}^2$ is the idealized null hypersurface $r = \infty$ described by coordinates (u, x). The metric γ is written as follows:

$$\gamma = -UV du^2 - 2U du dr + r^2 h_{\alpha\beta} \left(dx^\alpha + W^\alpha du\right) \left(dx^\beta + W^\beta du\right).$$

If the asymptotic expansion of $h_{\alpha\beta}$ is

$$h_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{C_{\alpha\beta}(u, x)}{r} + O(r^{-2}),$$

the Bondi Energy Loss formula is given as:

$$\frac{\partial}{\partial u} E(u) = -\int_{\mathbb{S}^2} N_{\alpha\beta} N^{\alpha\beta} dA_{\mathbb{S}^2},$$

where $N_{\alpha\beta} := \partial_u C_{\alpha\beta}$.

Outlook

- We recover non-conservation of energy and linear momentum of AH IDS using the Einstein evolution equations
- This is obtained for a class of observers $\mathcal{H}$
- This can also be done in the presence of matter
- Next step: Evolution of charges associated to other KIDs (*work in progress*)

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Thank you!