Topology and singularities in cosmological spacetimes obeying the null energy condition: Rigidity aspects for prime initial data

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A singularity theorem by Galloway Ling

Theorem (Galloway, Ling, 17)

(Thm. 1 in [1])

1. *Suppose V is a smooth, compact 3-dim. Riemannian manifold.*
2. *Let M be a (3+1) dimensional globally hyperbolic spacetime, with spacelike Cauchy surface V, which satisfies the NEC.*
3. *Assume V is expanding in all directions, i.e. the future second fundamental form (defined w.r.t. the future normal) is positive definite.*

Then either

1. *V is a spherical space or*
2. *M is past null geodesically incomplete.*

Expanding the previous theorem

Theorem (Ling, R., Simon, Steinbauer, 25)

1. *Suppose V is a 3-dim. smooth, compact, orientable, prime manifold.*
2. *Let M be a (3+1) dimensional globally hyperbolic spacetime, with spacelike Cauchy surface V, which satisfies the NEC.*
3. *Assume V is non-contracting in all directions, i.e. the future second fundamental form (defined w.r.t. the future normal) is positive semidefinite.*

Then either

1. *V is a spherical space*
2. *V is diffeomorphic to a surface bundle over the circle S^1 where the fibers V are orientable and totally geodesic MOTS (as embeddings in M).*
3. *There exists a finite cover $p : \tilde{V} \to V$ such that $\tilde{V}$ is as in (ii).*
4. *M is past null geodesically incomplete.*

Sketch of proof

- Note that the trace of the null second fundamental form can be written as $\theta^{\pm} = -tr_{\Sigma} K \pm H$.
- **Idea**: Find a minimal surface Σ in V that lifts to a noncompact cover $\tilde{V} \to V$ and apply Penrose in the covering manifold.

3-manifolds

Some important properties of compact 3-manifolds:

- Every compact, orientable 3-manifold can be written as a unique connected sum of prime manifolds, i.e. $V = V_1 \# ... \# V_k$.
- Prime manifolds are either spherical, diffeomorphic to $S^1 \times S^2$ or aspherical.
- A Haken manifold is an aspherical manifold which contains an embedded, incompressible surface.
- Every aspherical manifold is finitly covered by a Haken manifold [2].

The Haken case

Assume now that V is a Haken manifold in a complete spacetime.

- V contains a two-sided, incompressible minimal immersion $f : \Sigma \to V$ either embedded or double covering an embedded one-sided surface K [3][4].
- There exists a noncompact covering $p : \tilde{V} \to V$ such that f lifts to an embedding and $\tilde{V}$ is noncompact on both sideds of Σ.
- Using implicit function theorem to construct a local foliation around Σ we can apply Penrose in the covering to conclude that the fibers are totally geodesic MOTS.
- Either Σ is separating or nonseparating in V. One can then show that V is either a surface bundle over S^1 with fiber Σ or it has semibundle structure.

A sketch

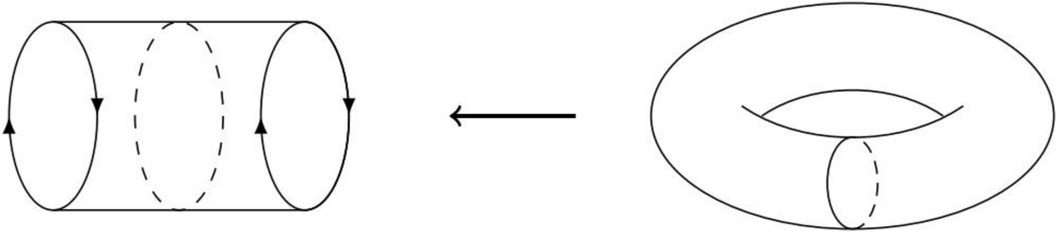

Figure: A simple sketch of a semibundle which is covered by a mapping torus

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