Stability and Instability of Relativistic Fluids in Slowly Expanding Spacetimes

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Der Wissenschaftsfonds.

Equations

Einstein-Euler-system:

$$\mathrm{Ric}[g]_{\mu\nu} - \frac{1}{2}\mathrm{R}[g]g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \qquad \text{(Einstein)}$$

$$\nabla_\alpha T^{\alpha\mu} = 0, \qquad \text{(rel. Euler)}$$

$$\rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu) = T_{\mu\nu}.$$

The relativistic Euler equations are equivalent to

$$u^\mu \nabla_\mu \rho + (\rho + p)\nabla_\mu u^\mu = 0,$$

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- $0 < K < \frac{1}{3}$: Massive fluids

Some solutions

FLRW spacetimes:

$$g = -dt^2 + a(t)^2 g_M,$$

where M is $\mathbb{H}^3, \mathbb{S}^3, \mathbb{R}^3$ of constant curvature κ. $a(t)$ is referred to as the **scale factor**. We will refer to $a(t) = t$ as **linear expansion**.
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3. Milne universe *(non-accelerated)*

$$\kappa = -1, \quad \Lambda = 0, \quad \rho = 0,$$
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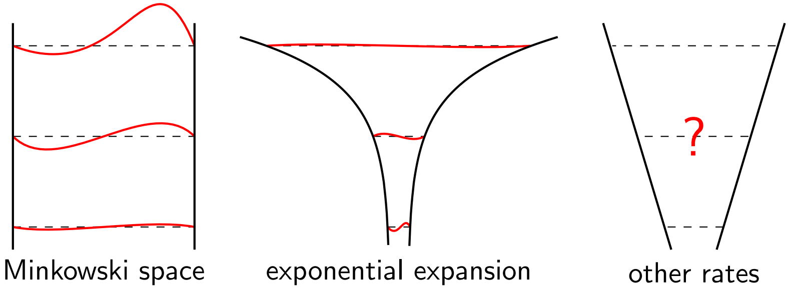

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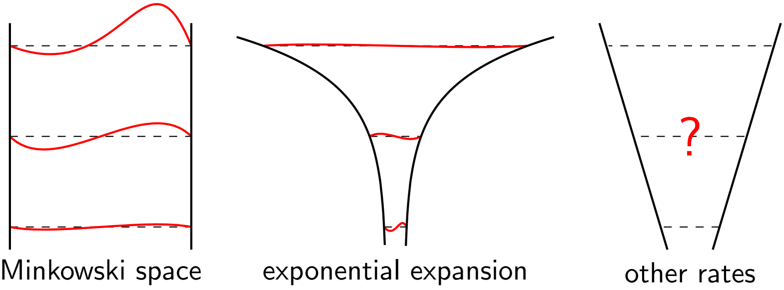

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Question: What expansion rate is sufficient (and for what fluid)?

Recent developments

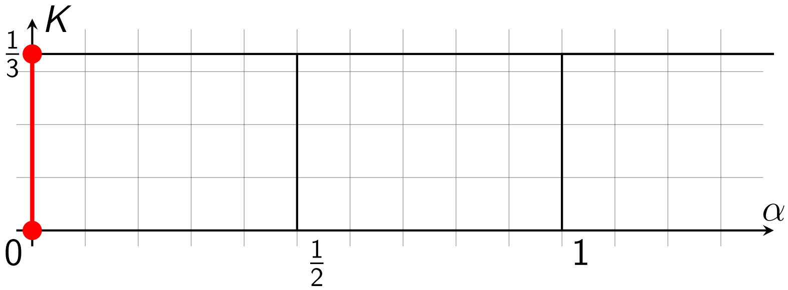
- [Christodoulou, 2007]
  - → Relativistic Euler on Minkowski space
  - → Singularities form in finite time
- [Brauer, Rendall, and Reula, 1994]
  - → Newtonian setting
  - → Fluid stabilization
- [Rodnianski and Speck, 2013]
  - → Euler-Einstein-system
  - → Exponential expansion rate
- [Speck, 2013]
  - → Relativistic Euler
  - → Various stability results for accelerated expansion ($a(t) > t$) and a sharp instability result for radiation at $a(t) = t$.
- Many more works by Hadžić, Oliynyk, Friedrich, Valiente-Kroon, ...

Stability in slow expansion

From now on: **Power law inflation**

$$g = -dt^2 + t^{2\alpha}\gamma.$$

[Speck, 2013],[Fajman, Oliynyk, and Wyatt, 2021],[Fajman, O, Oliynyk, and Wyatt, 2024]:



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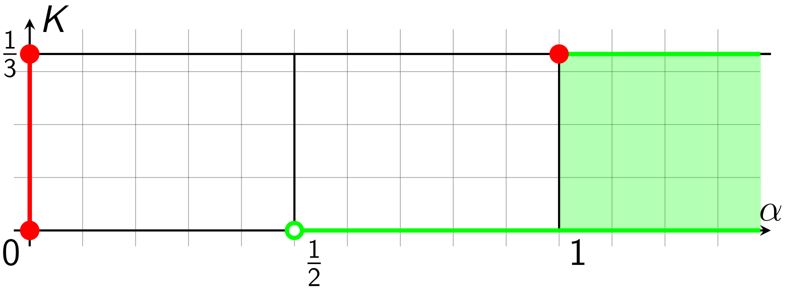

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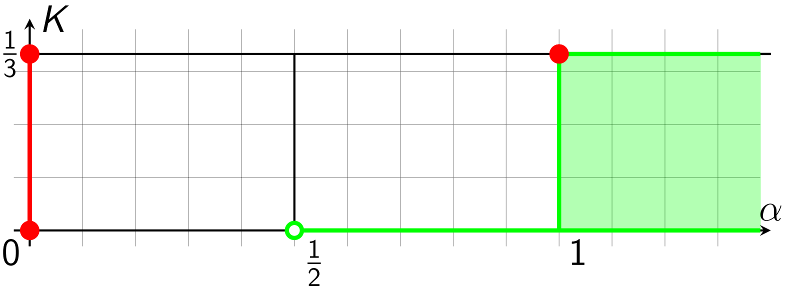

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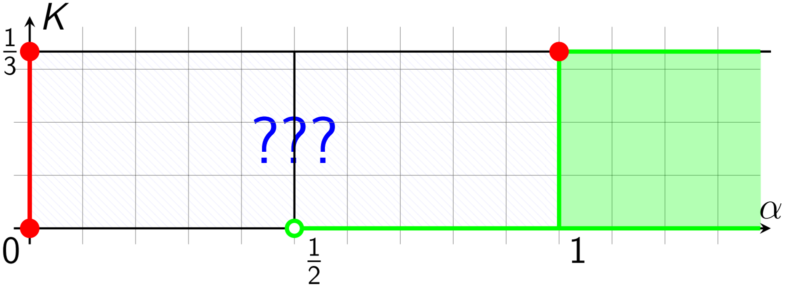

What happens in **decelerated regimes**?

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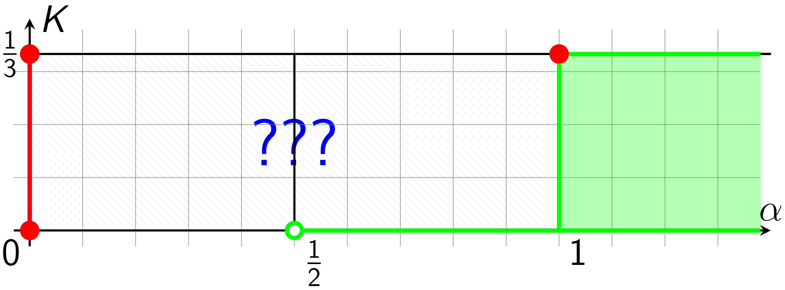

What happens in **decelerated regimes**?
Does stability depend non-trivially on K?

Decelerated regime: Analytical analysis

Background:

$$(\mathbb{R} \times \mathbb{T}^3, -dt^2 + t^{2\alpha}\delta_{ij}dx^i dx^j).$$

Equations of motion in $1+1$-symmetry

$$\begin{aligned}\partial_t v = &-\frac{\alpha(1-3K)}{t}v - t^{-\alpha}\frac{1-K}{1-Kv^2}v\partial_x v \\ &- t^{-\alpha}\frac{K}{1+K}\frac{(1-v^2)^2}{1-Kv^2}\partial_x L + \alpha(1-K)(1-3K)\frac{t^{-\alpha}}{1-Kv^2}\,v^3,\end{aligned}$$

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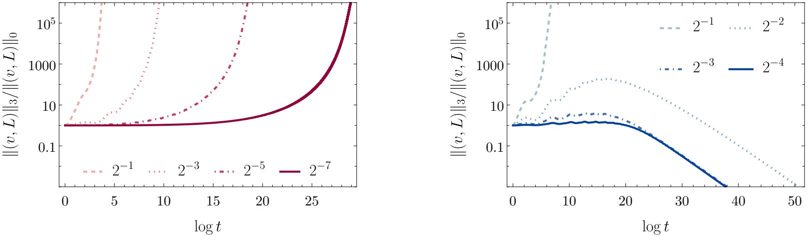
Condition for closing the estimate: $K < 1 - \frac{2}{3\alpha}$.

Decelerated Regime: Numerical analysis

Left: (α, K)=(0.7,1/6), Right: (α, K)=(0.9,1/6):



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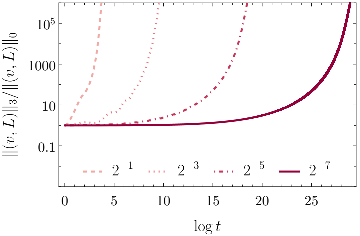
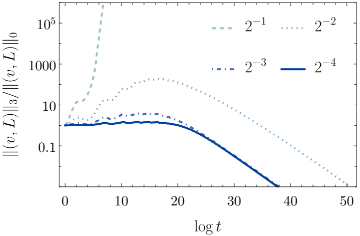
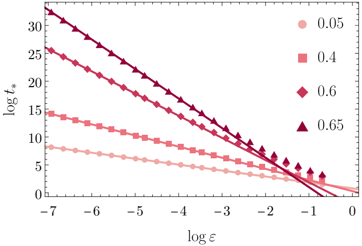
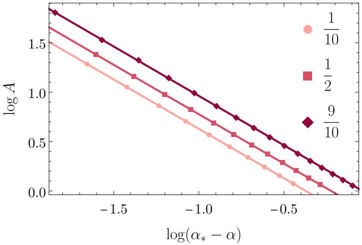


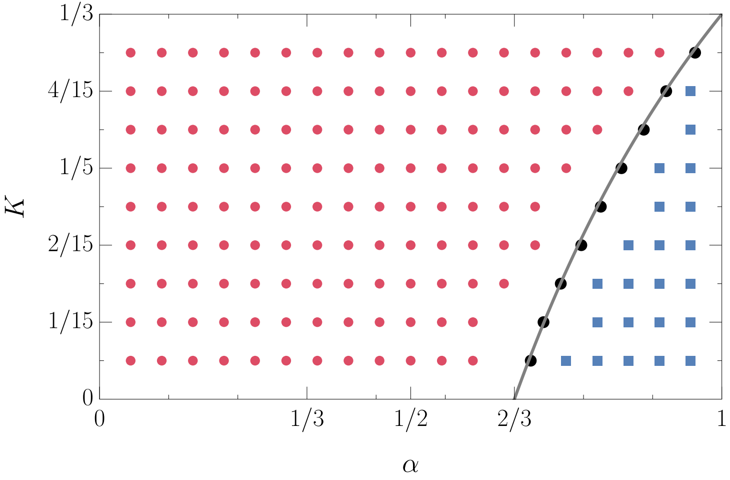


Left: $K = 1/6$:



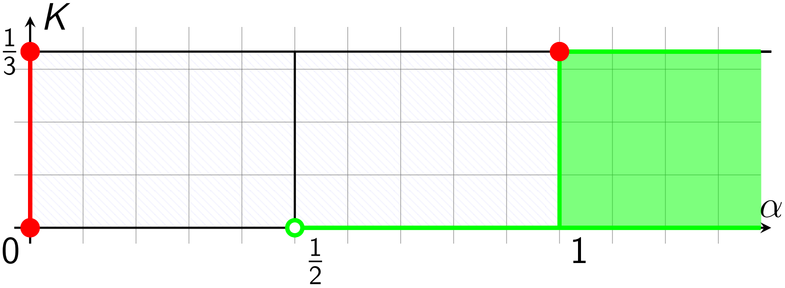


Comparison Analysis/Numerics



Results for the boundary cases

In addition to the bulk region $0 < K < \frac{1}{3}$, we have analytical results for dust and radiation:



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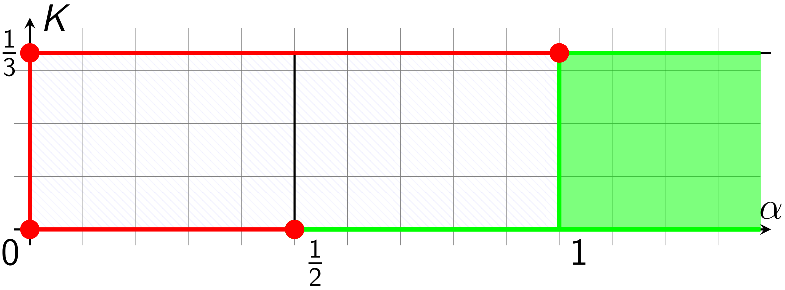

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Thank you for your attention!

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