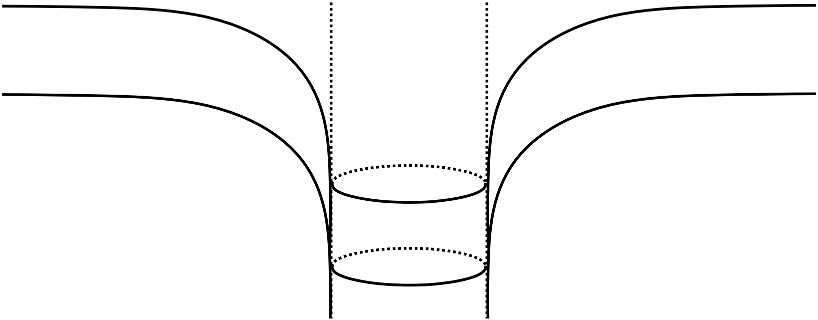
Uniform Temple charts and applications to null distance

Benjamin Meco,
joint with Anna Sakovich and Christina Sormani

Convergence of spacetimes



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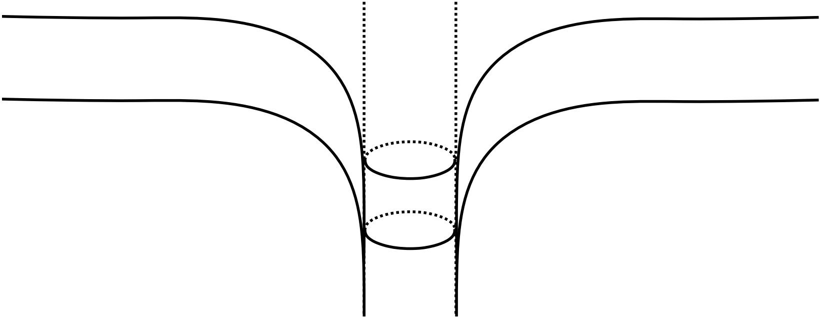

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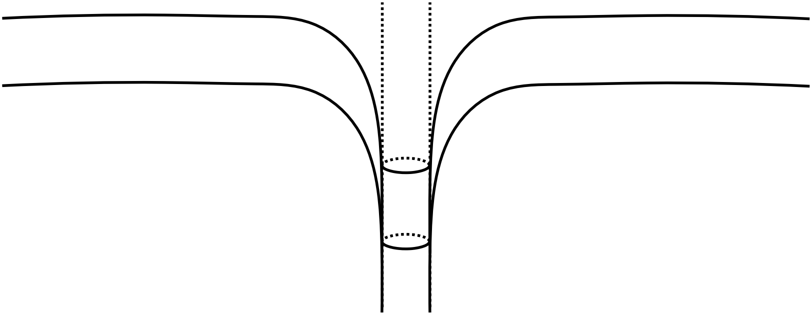

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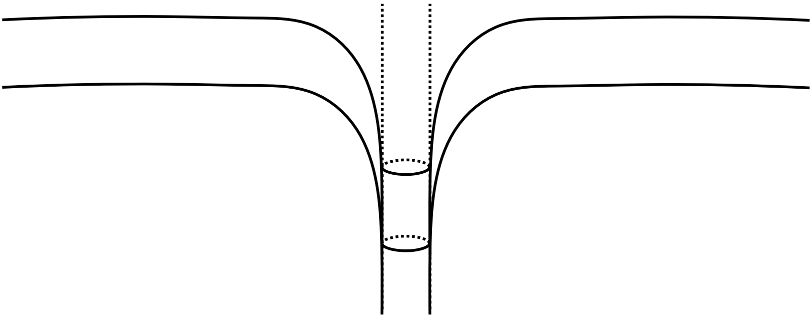

We would like a notion of distance between spacetimes that applies when spacetimes are not diffeomorphic and their metric tensors are not close in a smooth sense.

Why do we need a metric theory of spacetimes?

Other questions one could try to answer:

- Is our spacetime well-approximated by cosmological models?
- Can we compare spacetimes with different topology?
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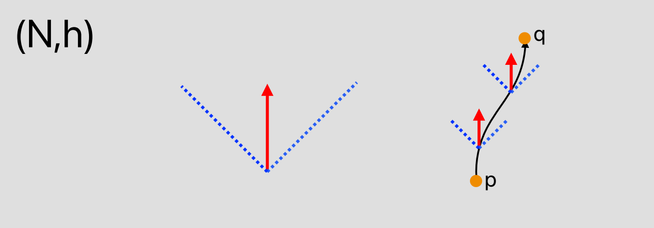
Problem: In contrast to Riemannian manifolds spacetimes are not natural metric spaces.

Spacetimes and causal structure



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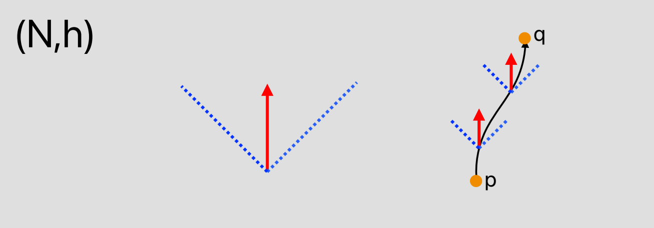

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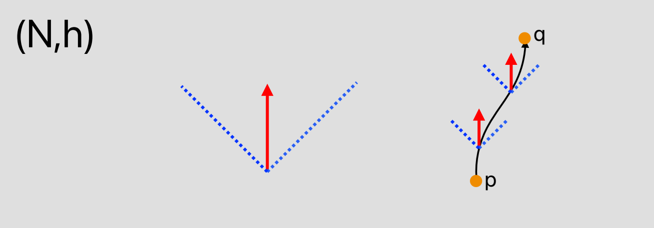

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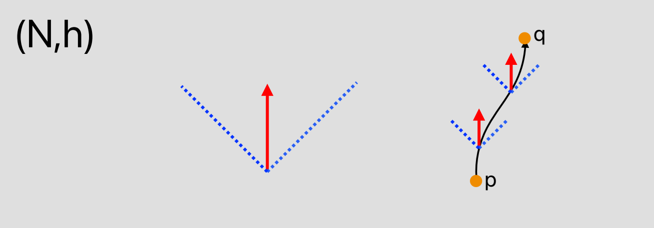

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- A curve $\gamma : I \to N$ is causal if $\dot{\gamma}$ is always causal.
- We denote the causal future of a point $p \in N$ by $J^+(p)$.

Time functions

Definition (Time functions)

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Example: A "canonical" time function is the cosmological time function of Andersson-Galloway-Howard 1997.

$$\tau_g(q) := \sup_{\substack{\gamma:[0,1]\to N \\ \gamma \text{ future causal} \\ \gamma(1)=q}} \int_0^1 \sqrt{-g(\dot\gamma(s),\dot\gamma(s))}ds.$$

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Could be infinite, a **regular** cosmological time function takes values in $(0,\infty)$.

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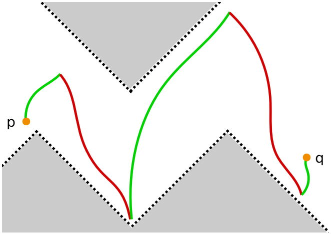
where γ is a piecewise causal curve from p to q, with breakpoints at each $s_1, \ldots, s_{k-1}$.



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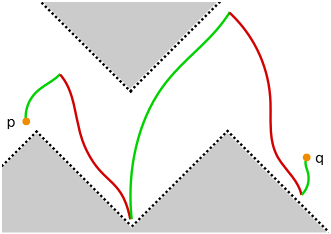
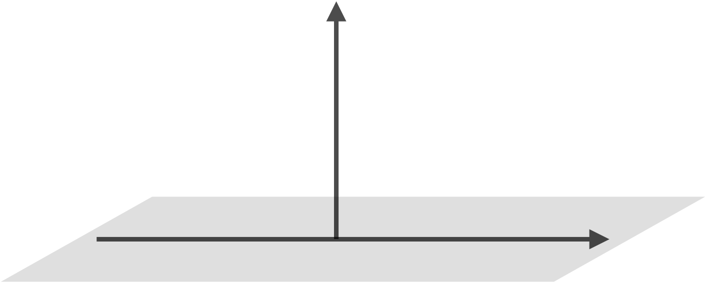


If τ is **locally anti-Lipschitz** (Chruściel-Grant-Minguzzi 2016), e.g. $\tau = \tau_g$ is a regular cosmological time function, then $\hat{d}_\tau$ is definite.

Example: Null distance in Minkowski space, $\tau = t$



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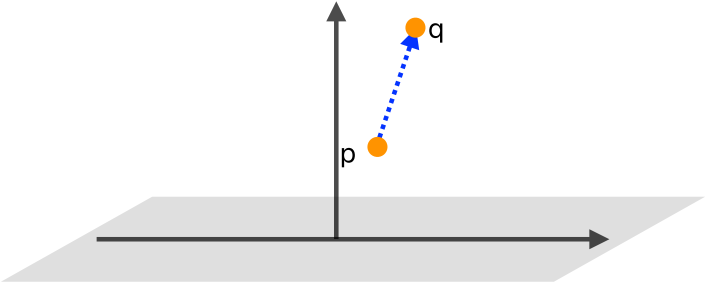

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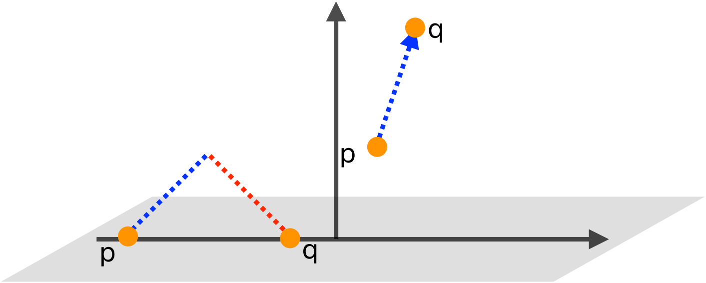

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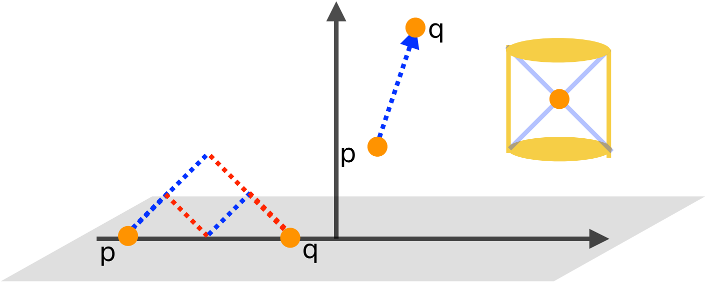

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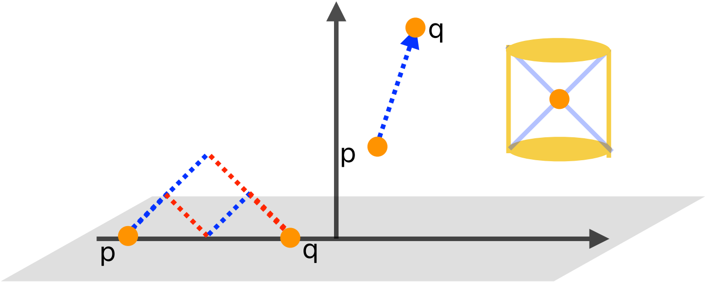

The null distance in this case is:

$$\hat{d}_t((t_1, \vec{x}_1), (t_2, \vec{x}_2)) = \max(|t_1 - t_2|, |\vec{x}_1 - \vec{x}_2|).$$

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We want our notion of distance to satisfy:

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i.e. spacetimes are "the same" iff there is a Lorentzian isometry between them.

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Consider the following subsets of $(\mathbb{R}^2, -dt^2 + dx^2)$, $\tau(t, x) = t$.

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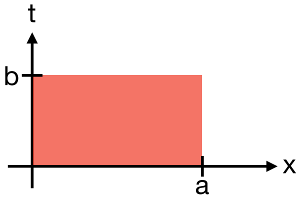

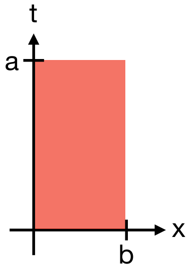

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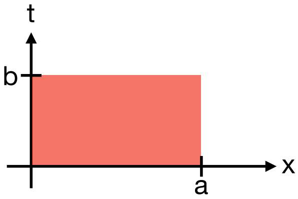

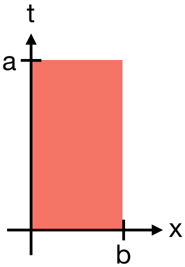

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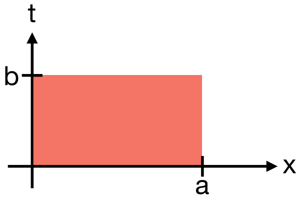

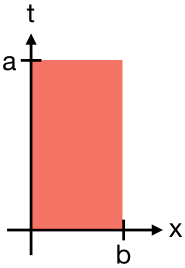

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The map $F(t,x) = (x,t)$ is a metric isometry, but these spacetimes are not isometric as Lorentzian manifolds.

Encoding causality

We say that τ and $\hat{d}_\tau$ **encode causality** if for all $p, q \in N$:

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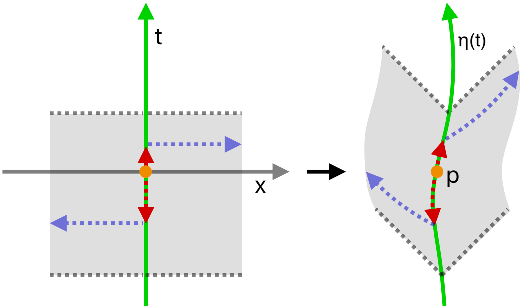
- Galloway 2023: τ with **future causally complete level sets**.

Temple charts



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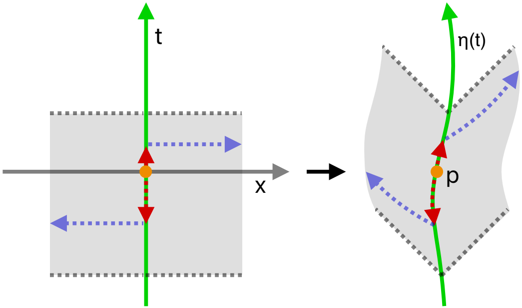

$$\Phi_p(t, x^1, \ldots, x^n) = \exp_{\eta(t)}\left(|\vec{x}|e_0 + \sum_{i=1}^{n} x^i e_i\right), \quad \omega_p(\Phi_p(t, \vec{x})) := t,$$

where $\{e_0, \ldots, e_n\}$ are frame fields and η is a future timelike geodesic with $\eta(0) = p$ and $\dot{\eta} = e_0$.

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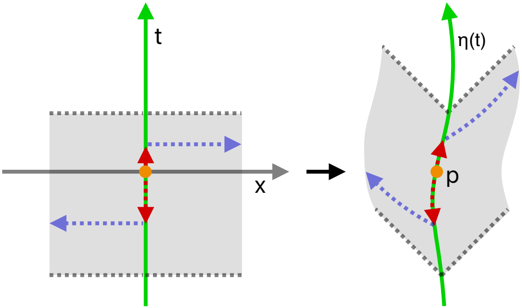

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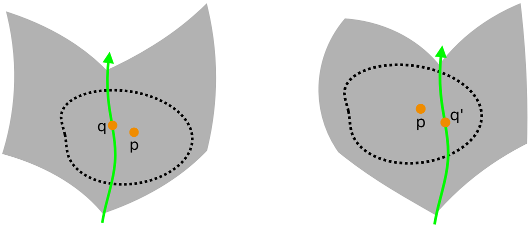
where $\{e_0, \ldots, e_n\}$ are frame fields and η is a future timelike geodesic with $\eta(0) = p$ and $\dot{\eta} = e_0$. We have:

$$q \in J^+(p) \iff \omega_p(q) \geq 0.$$

Uniform Temple charts

Theorem (M.-Sakovich-Sormani, TBP)

Every $p \in N$ has a neighborhood U_p such that for all $q \in U_p$, the image of the Temple chart Φ_q covers U_p.



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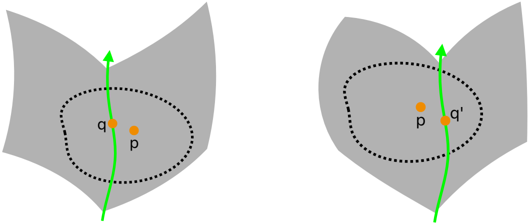

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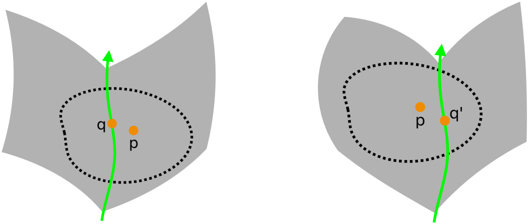

Moreover, there is a Riemannian metric, g_R, on U_p such that

$$\text{for all } q, q' \in U_p\text{: } K^{-1} d_{g_R}(q, q') \le \hat{d}_\tau(q, q') \le K d_{g_R}(q, q').$$

An isometry theorem

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Let (N_1^{n+1}, g_1, τ_1) *and* (N_2^{n+1}, g_2, τ_2)*,* $n \geq 2$*, be Lorentzian manifolds equipped with Lipschitz time functions* τ_i *such that*

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Recall: The causal structure determines the Lorentzian metric up to a conformal factor (Hawking et al.).

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$$g_i(\nabla\tau_i, \nabla\tau_i) = -1 \quad \textit{almost everywhere}, \quad i = 1, 2.$$

A bijection $F : N_1 \to N_2$ *such that*

$$\forall p, q \in N_1 : \quad \hat{d}_{\tau_1}(p, q) = \hat{d}_{\tau_2}(F(p), F(q)), \quad \tau_1(p) = \tau_2(F(p)),$$

is a diffeomorphism and a Lorentzian isometry.

Recall: The causal structure determines the Lorentzian metric up to a conformal factor (Hawking et al.).

Remark: This was proven in Sakovich-Sormani 2022 under an extra assumption that causality is globally encoded by $\hat{d}_{\tau_i}$ and τ_i.

Conclusions

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Sakovich and Sormani 2024 discuss several possible definitions of (definite) distances between spacetimes, based on τ and $\hat{d}_\tau$.

Thank you for listening!