Two spherically symmetric matter models near the weak null singularity in an Einstein-Maxwell-scalar field spacetime

CERS15 Short Talk

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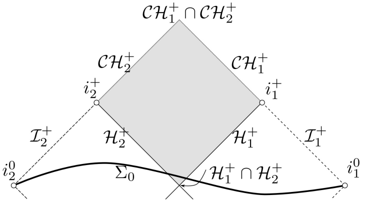
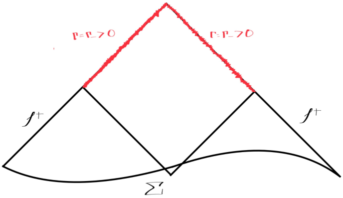
Strong Cosmic Censorship in General Relativity

SCC (C^2 formulation), Christodolou [1]

Maximal globally hyperbolic future developments for the Einstein Field Equations corresponding to ***generic asymptotically flat*** *initial data are future-****inextendible*** *as time-oriented Lorentzian manifolds* ***with*** *C^2* ***metrics****.*

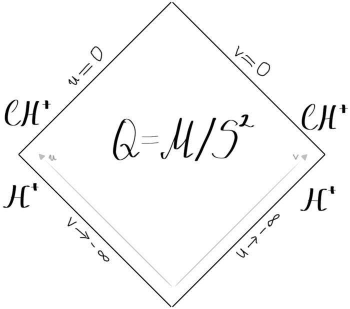




- Scalar field perturbation of RN $\implies$ the Cauchy horizon is a WNS $\implies$ does not violate SCC [2, 4].

The Einstein-Maxwell scalar field class of spacetimes

- $\mathcal{M} = \mathcal{Q} \times S^2 = (-\infty, 0)^2 \times S^2$, $g = -e^{2\omega} dudv + r^2(u,v) d\Omega_2^2$
- e^{ω} and r are smooth and positive in $\mathcal{Q}$ and extend continuously to positive functions on $\mathcal{CH}^+$



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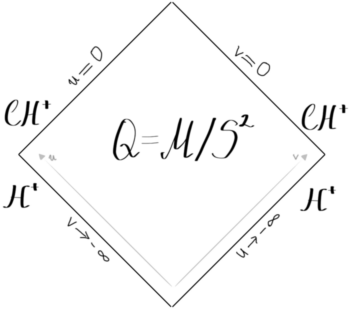
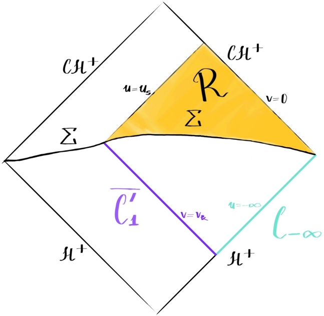
In the region of interest $\mathcal{R} = \mathcal{Q}_{\mathcal{R}} \times S^2 \subset \mathcal{M}$:

- $r_{,v} \to -\infty$ as $v \to 0^-$ for all $u \in (-\infty, u_s)$ $(\mathcal{CH}^+ = \{v = 0\} = WNS)$
- $r_{,u} < 0$ and $r_{,v} < 0$ in $\mathcal{R}$.
- Important L^1 bounds of geometric quantities hold **specifically** in $\mathcal{R}$.



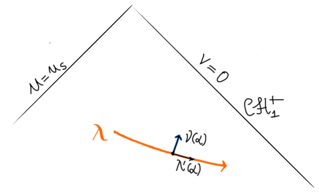


Spherical dust. Preliminaries for Theorem 1

Definition ((n_0, N_0, C_K)-admissible curve)

Let $\lambda : (-a, a) \to \mathcal{Q}_{\mathcal{R}}$ be a (nonconstant) smooth spacelike curve. Let $0 < n_0 < N_0, C_K > 0$. Say λ is an (n_0, N_0, C_K)**-admissible curve** if $\forall \alpha \in (-a, a)$

1. $u[\lambda(\alpha)] > -\infty$
2. $n_0 < |(\lambda')^u|, |(\lambda')^v| < N_0$ *($\lambda' \equiv$ tangent to λ)*
3. 'Curvature of λ' $= \left| \frac{\langle \lambda', \nabla_{\lambda'} \nu \rangle}{\langle \lambda', \lambda' \rangle} \right| \leq C_K$ *($\nu(\alpha) \equiv$ f-d unit normal to λ)*
4. $v[\lambda(\alpha)] < \frac{u_s - u[\lambda(\alpha)]}{C_g} \times \left(- \frac{(\lambda')^v(\alpha)}{(\lambda')^u(\alpha)} \right)$



Definition (Geodesic variation based on λ)

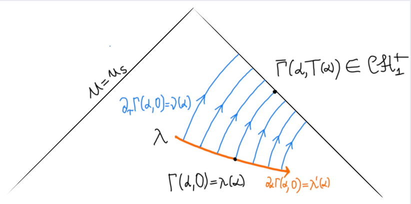
Given a (n_0, N_0, C_K)-admissible curve λ define **the geodesic variation based on** λ to be the map $\overline{\Gamma}$, given by

$$\overline{\Gamma}(\alpha,\tau) := \exp_{\lambda(\alpha)}(\tau\nu(\alpha)) \tag{1}$$

$$\text{on } D = \{(\alpha,\tau)\in\mathbb{R}^2 : -a<\alpha<a;\ 0\le\tau\le T(\alpha)\} \tag{2}$$

where $\alpha \mapsto T(\alpha)$ is defined by the condition $v\left[\overline{\Gamma}(\alpha, T(\alpha))\right] = 0$.

Denote by D^0 be the interior of D and by $\Gamma := \overline{\Gamma}|_{D^0}$ the restriction of $\overline{\Gamma}$ to D^0.

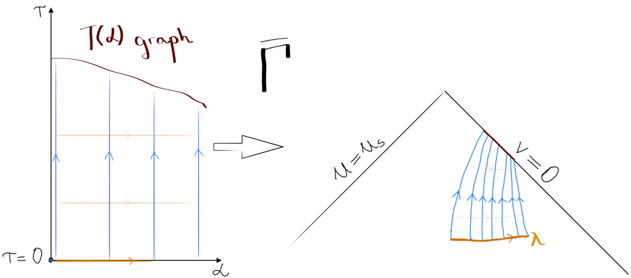


Spherical dust. Statement of Theorem 1

Theorem 1 (Regularity of Spherical Dust)

Given a (n_0, N_0, C_K)-admissible curve λ, let $\overline{\Gamma} : D \to \overline{\Gamma}(D) \subset \overline{\mathcal{Q}_{\mathcal{R}}}$ be the geodesic variation based on λ. Then there exists a small enough v_{min} depending on C_K, n_0 and N_0 such that if $\lambda^v(\alpha) > v_{min}$ for all $\alpha \in (-a, a)$ then

1. $\Gamma : D^0 \to Image(\Gamma) \subset \mathcal{Q}_{\mathcal{R}}$ *is a* ***diffeomorphism***.
2. $\overline{\Gamma} : D \to Image(\overline{\Gamma}) \subset \overline{\mathcal{Q}_{\mathcal{R}}}$ *is a* ***homeomorphism***.



Spherical dust. Statement of Theorem 1

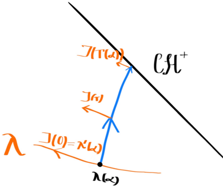
Theorem 1 (Regularity of Spherical Dust), continued

Consider ***spherically symmetric dust*** *in* $\mathcal{R}$ *with 4-velocity vector field* $U(u,v) = \partial_\tau \Gamma \circ \Gamma^{-1}(u,v)$ *at all points in* $Image(\Gamma) \subset \mathcal{Q}_{\mathcal{R}}$. *Let* $\rho = \rho(u,v)$ *to be the energy density of the dust on* $Image(\Gamma) \subset \mathcal{Q}_{\mathcal{R}}$.

Assume that ρ *is* ***smooth and positive*** *on* $Image(\lambda)$ *and* ***extends to a positive function*** *on* $\overline{Image(\lambda)}$, *and that* U *and* ρ *satisfy the* ***transport equation*** *on* $\Gamma(D^0)$. *Then*

3. $\exists C > 0$, *depending on* $\|\rho\|_{C^0(\lambda)}$, n_0, N_0, v_{min} *and* C_g *such that* $\rho(u,v) \leq C$ *in* $Image(\overline{\Gamma})$.

1. Prove that the **Jacobi field** along an infalling radial timelike geodesic based λ **stays bounded and bounded away from zero**.



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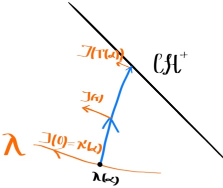
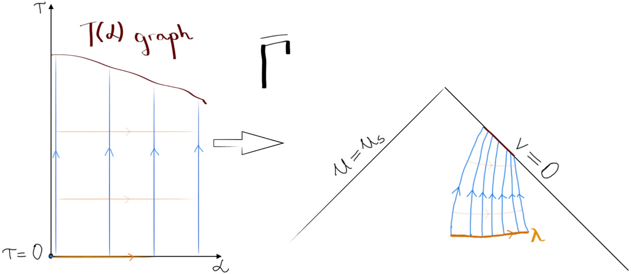
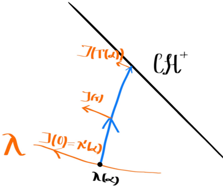
2. Prove $\Gamma : D^0 \to \text{Image}(\Gamma) \subset \mathcal{Q}_{\mathcal{R}}$ is locally a diffeomorphism in the interior & injective. Conclude Γ **is a diffeomorphism onto its image**.
3. Prove that $\alpha \mapsto \bar{\Gamma}(\alpha, T(\alpha)) \in \mathcal{CH}^+$ is injective and depends continuously on α. Conclude that $\bar{\Gamma}$ is a **homeomorsphism onto its image**





Spherical dust. Key steps in the proof.



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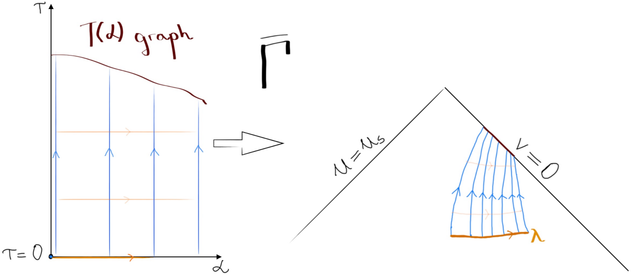
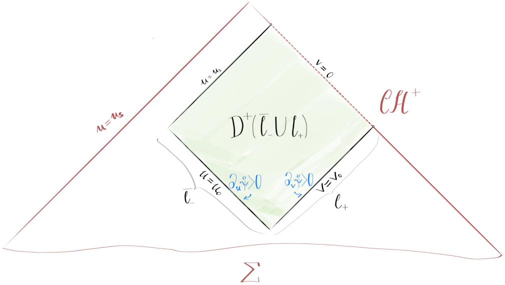


4. **Integrate the tranpsort equation $\to$ prove that ρ is bounded.**

Scalar field characteristic IVP. Preliminaries for Theorem 2.

- **Initial hypersurface.** Let $(u_0, v_0) \in \mathcal{Q}_{\mathcal{R}}$. Define $\mathcal{C} := \mathcal{C}_+ \cup \overline{\mathcal{C}_-}$ in $\mathcal{Q}_{\mathcal{R}}$, where $\mathcal{C}_+ = \{u_0\} \times [v_0, 0)$; and $\overline{\mathcal{C}_-} = [u_0, u_s] \times \{v_0\}$



- **Spherically symmetric homogeneous linear wave equation with characteristic initial data**:

$$\Box_g \psi(u, v) = 0 \quad \in D^+(\mathcal{C}), \text{ in coordinates } \iff$$

$$\partial_u \partial_v \psi(u, v) = -(\ln r)_{,u} \psi_{,v} - (\ln r)_{,v} \psi_{,u} \in D^+(\mathcal{C}) \tag{3}$$

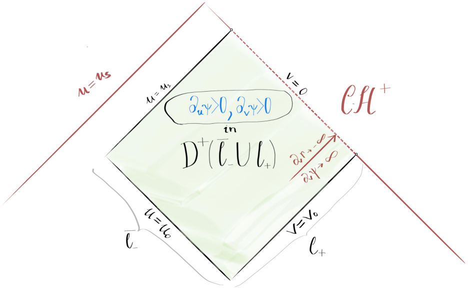
$$\psi|_{\mathcal{C}} = \mathring{\psi}, \text{ where } \mathring{\psi}|_{\mathcal{C}_+} \in C^2(\mathcal{C}_+) \text{ and } \mathring{\psi}|_{\overline{\mathcal{C}_-}} \in C^2(\overline{\mathcal{C}}_-) \tag{4}$$

Scalar field characteristic IVP. Statement of Theorem 2.

Theorem 2 (Wave equation blow-up on $\mathcal{CH}^+$)

Let ψ be a C^2 solution of the characteristic initial value problem for the wave equation. Assume that the null derivatives of the initial data satisfy the ***monotonicity assumptions*** $\partial_u \mathring{\psi}|_{\overline{\mathcal{C}_-}} > 0$ *and* $\partial_v \mathring{\psi}|_{\mathcal{C}_+} > 0$. *Then*

$$\partial_v \psi(u, v) \gtrsim \inf_{\mathcal{C}_-} (\partial_u \mathring{\psi}) \int_{u_0}^{u} (-r_{,v})(u', v) du' \tag{5}$$



Scalar field characteristic IVP. Steps to prove Theorem 2.

1. Prove that **monotinicity is propagated in** $D(\mathcal{C})$ i.e. $\partial_u\psi > 0, \partial_v\psi > 0$ in $D(\mathcal{C})$. *(bootstrap argument similar to [3])*

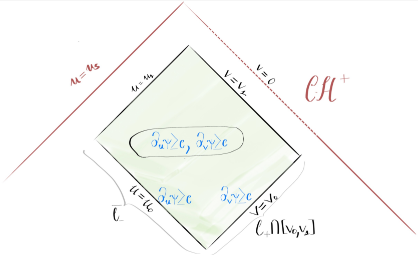
Proposition 3 (Monotonicity is propagated in the interior).

Let $v_1 \in (v_0/2, 0)$ be arbitrary and let

$$c := \min\{\inf_{\mathcal{C}_-} \partial_u \mathring{\psi}, \inf_{\mathcal{C}_+ \cap [v_0, v_1]} \partial_v \mathring{\psi}\} > 0 \qquad (6)$$

Under the hypotheses of Theorem 2,

$$\partial_u\psi(u,v) \geq c \text{ and } \partial_v\psi(u,v) \geq c \text{ in } [u_0, u_1] \times [v_0, v_1].$$



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Under the hypotheses of Theorem 2,
$\partial_u\psi(u,v) \geq c$ and $\partial_v\psi(u,v) \geq c$ at every $(u,v) \in [u_0,u_1] \times [v_0,v_1]$.

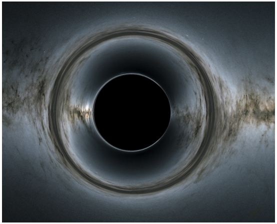
2. **Integrate the wave equation** to obtain (5).
3. Since $\mathcal{CH}^+$ is a WNS, $\partial_v r \to -\infty$, so $\partial_v\psi \to \infty$.

Conclusion

Under reasonable assumptions, the two matter models show wildly different behaviour near the WNS:

- For admissible initial data, (ρ, U) stay **regular** in the following sense: **no crossing** of fluid trajectories, **no blow-up** of energy density.
- Under monotonicity assumption on the initial data the solution of the wave equation **blows up** in C^1: $\partial_v \psi \xrightarrow{v \to 0} \infty$.

Thank you for listening!



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