# How Robinson's method to prove Black Hole uniqueness influences Geometric Analysis


Ariadna León Quirós

Joint work with C. Cederbaum

Department of Geometric Analysis, Differential Geometry and Relativity Theory

University of Tübingen


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**Definition (Hawking mass)**

Let $(M, g)$ be a 3-Riemannian manifold embedded in a spacetime $(\mathscr{L}, \bar{g})$, and let $\Sigma \subset M$ be a closed 2-surface. Then, we define the Hawking mass as

$$\mathrm{m_H}(\Sigma) = \frac{1}{4\pi}\sqrt{\frac{|\Sigma|}{16\pi}}\left(4\pi - \frac{1}{4}\int_\Sigma \mathrm{H}^2 \mathrm{d}\sigma\right), \tag{1}$$

where $\frac{1}{4}\int_\Sigma \mathrm{H}^2\mathrm{d}\sigma$ is known as the **Willmore energy**.

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**Theorem (Willmore-type inequality)**

*Let $(M, g)$ be a complete, non-compact, Riemannian n-manifold with Ric $\geq 0$ and Euclidean Volume Growth. If $\Omega \subset M$ is bounded and open subset with smooth boundary, then*

$$\int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma \geq AVR(g)|\mathbb{S}^{n-1}| \tag{2}$$

*where $AVR(g) \in (0, 1]$ is the asymptotic volume ratio of $(M, g)$. Moreover, the equality holds if and only if $(M \setminus \Omega, g)$ is isometric to the exterior of a sphere in a round cone.*

## Corollary

*This is a generalization of the Willmore inequality in Euclidean space*

$$\int_{\partial\Omega} \left| \frac{H}{n-1} \right|^{n-1} d\sigma \geq |\mathbb{S}^{n-1}|,$$

*with equality if and only if* $\Omega$ *is a round ball.*

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## Corollary

*Considering the Asymptotically Euclidean case of the Willmore inequality, i.e.* $AVR = 1$*, for 3-manifolds excluding round cones, then*

$$m_H(\Sigma) < 0.$$

*And since in a large sphere limit the Hawking mass converges to the ADM energy, then*

$$E_{ADM} < 0.$$

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  - More transparent.
- The functionals of both proofs are related.

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- This paper also conveys the proof of the uniqueness of photon surfaces, and this can also be seen in Cederbaum-Cogo-Fehrenbach'24.
- This is also the method employed by Cederbaum-Miehe'24 to prove the Eclidean Willmore inequality aforementioned.
- Ongoing work to prove it when using $p$-harmonic functions for the Minkowski inequality by Babisch-Cederbaum.

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- We are dealing with the following boundary value problem, which is considered by potential theory by Agostiniani-Fogagnolo-Mazzieri.

$$\begin{cases} \Delta u = 0 & \text{in } M \setminus \bar{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u \to 0 & \text{as } |x| \to +\infty \end{cases} \tag{3}$$

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- We calculate its divergence, finding:
  - A coupled ODES for the functions $F(u)$ and $G(u)$, and
  - it is non-negative.

# Sketch of the Proof II

- We apply the divergence theorem, check integrability across critical and due to the non-negativity of the divergence of the vector field we can obtain, for $c + d \geq 0$ and $d \geq 0$,

$$d(n-2)^{\beta+1} \mathrm{AVR}(g)^{\frac{\beta}{n-2}} \mathrm{Cap}(\Omega)^{\frac{n-2-\beta}{n-2}} |\mathbb{S}^{n-1}| \leq \beta(c+d) \int_{\partial\Omega} |\nabla u|_g^{\beta} \mathrm{H} \, \mathrm{d}\sigma + \left( -\frac{n-1}{n-2}\beta(c+d) + d \right) \int_{\partial\Omega} |\nabla u|_g^{\beta+1} \mathrm{d}\sigma \tag{4}$$

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- Applying standard methods we obtain the generalized Willmore inequality

# Relation to Potential Theory

- By the method of Potential Theory of Agostiniani-Fogagnolo-Mazzieri'19, they prove the Willmore inequality by studying the monotonicity of the following functional $U_\beta : (0,1] \to \mathbb{R}$ defined as

$$U_\beta(t) = t^{-\beta\frac{n-1}{n-2}} \int_{\{u=t\}} |\nabla u|_g^{\beta+1} \mathrm{d}\sigma, \tag{5}$$

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- Define the following functional which depends on the vector field $X$ from Robinson's method,

$$\begin{aligned} \mathscr{H}_\beta^{c,d}(t) &= \int_{\{u=t\}} \langle X, \nu \rangle \mathrm{d}\sigma \\ &= \int_{\{u=t\}} \beta F(u) |\nabla u|_g^\beta \mathrm{H} + G(u) |\nabla u|_g^{\beta+1} \mathrm{d}\sigma \end{aligned} \tag{6}$$

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- Relation between the functionals,

$$\mathscr{H}_\beta^{1,0} = U_\beta'$$

# Outlook

- Potential theory allows us to prove **Hamilton's conjecture**,

**Theorem (Hamilton's Pinching Conjecture)**

*Let $(M, g)$ be a complete, connected, non-compact Riemannian Ricci–pinched 3–manifold. Suppose that $(M, g)$ has Euclidean volume growth, then it is flat.*

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- This was previously proved by Ricci flow (Chen-Zhu'00, Deruelle- Schulze-Simon' 22, Lott'24, and Lee-Topping'24 ), by Inverse Mean Curvature Flow (Huisken-Körber'24) and by Potential Theory (Benatti-Mantegazza-Oronzio-Pluda'24 and in my master thesis).

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- Future projects:
  - Proof of Positive Mass Theorem.
  - Proof of Penrose Inequality.

Thank you for your attention!

## Theorem (Divergence Inequality)

*Let u be a solution to 3. Then, on $M \setminus (\bar{\Omega} \cap Critu)$, the following*

$$div_g \left( F(u)\nabla|\nabla u|_g^\beta + G(u)|\nabla u|_g^\beta \nabla u \right) |\nabla u|_g^2$$
$$\geq a_\beta F(u) \left| \nabla|\nabla u|_g^2 - \frac{2(n-1)}{(n-2)u}|\nabla u|_g^2 \nabla u \right|_g^2 |\nabla u|_g^{\beta-2} \tag{7}$$

*holds with the constant $a_\beta = \frac{\beta}{4}\left(\beta - \frac{n-2}{n-1}\right)$ for $\beta \geq 0$ and the functions*

$$F(u) = cu^{-\beta\frac{n-1}{n-2}+2} + du^{-\beta\frac{n-1}{n-2}+1},$$
$$G(u) = -\beta\frac{n-1}{(n-2)u}F(u) + du^{-\beta\frac{n-1}{n-2}},$$

*where $c, d \in \mathbb{R}$.*

# Proof of the Divergence Inequality Theorem

- Calculate the divergence.

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- Apply Bochner formula for general Riemannian metric and apply the reduced Kato's identity

$$\Delta_g |\nabla u|_g^2 - \langle \nabla |\nabla u|_g^2, \nabla u \rangle_g = 2 \left( \mathrm{Ric}(\nabla u, \nabla u) + |\nabla \nabla u|_g^2 \right),$$
$$|\nabla^2 u|_g^2 \geq \frac{n}{n-1} |\nabla |\nabla u|_g|_g^2,$$

and also use the fact that Ric $\geq 0$.

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$$|\nabla^2 u|_g^2 \geq \frac{n}{n-1} |\nabla |\nabla u|_g|_g^2,$$

  and also use the fact that $\mathrm{Ric} \geq 0$.
- Calculate the square of the RHS of 7 and compare the coefficients gives coupled ODEs for $F(u)$ and $G(u)$, and solve the ODEs.

## Theorem (Parametric geometric inequality)

*Let $u$ be a solution to (3) with $\Omega$ bounded domain with smooth boundary. For $c, d \in \mathbb{R}$ such that $F(u)$ given before and $\beta \geq \frac{n-2}{n-1}$, we get*

$$d(n-2)^{\beta+1} AVR(g)^{\frac{\beta}{n-2}} Cap(\Omega)^{\frac{n-2-\beta}{n-2}} |\mathbb{S}^{n-1}| \leq \beta(c+d) \int_{\partial\Omega} |\nabla u|_g^{\beta} H d\sigma \tag{8}$$

$$+ \left( -\frac{n-1}{n-2} \beta(c+d) + d \right) \int_{\partial\Omega} |\nabla u|_g^{\beta+1} d\sigma \tag{9}$$

*Moreover, the equality holds if and only if $(M \setminus \Omega, g)$ is isometric to*

$$\left( [r_0, +\infty) \times \partial\Omega, dr \otimes dr + (r/r_0)^2 g_{\partial\Omega} \right), \quad \text{with } r_0 = \left( \frac{|\partial\Omega|}{AVR(g)|\mathbb{S}^{n-1}|} \right). \tag{10}$$

*In particular, $\partial\Omega$ is a connected totally umbilic submanifold with constant mean curvature.*

- Apply the divergence theorem in the left hand side of the just obtained divergence inequality (7), for $c, d$ s.t. $F(u) \geq 0$ and $\beta \geq \frac{n-2}{n-1}$, thus obtaining

$$0 \leq \int_{\{u<1\}} div_g Z d\mu = \int_{\partial\Omega} \beta F(u)|\nabla u|_g^{\beta} H + G(u)|\nabla u|_g^{\beta+1} d\sigma \tag{11}$$

$$- \lim_{t\to 0} \int_{\{u=t\}} \beta F(u)|\nabla u|_g^{\beta} H + G(u)|\nabla u|_g^{\beta+1} d\sigma. \tag{12}$$

- Evaluate the first term.
- Calculate the second term using the asymptotic for $u$,

$$\lim_{|x|\to+\infty} \frac{u(x)}{|x|^{2-n}} = \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \tag{13}$$

and the definition of electrostatic capacity of $\Omega$ as

$$\mathrm{Cap}(\Omega) = \inf\{\frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_M |\nabla u|^2 d\mu \mid u \in \mathscr{C}_c^{\infty},\ u = 1 \text{ in } \Omega\} \tag{14}$$

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- Then we apply the following Corollary.

**Corollary ($L^p$-norm of the normal derivative)**

*If $u$ solves (3), then we have that*

$$\|\nabla u\|_{L^p(\partial\Omega)} \leq \frac{n-2}{n-1}\|H\|_{L^p(\partial\Omega)} \tag{15}$$

*with equality in the rigidity case*