Oscillatory spacelike singularities: The Bianchi type $\mathrm{VI}_{-1/9}$ vacuum models

Phillipo Lappicy

Universidad Complutense de Madrid

with **Claes Uggla** (Karstads Universitet)

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Introduction

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- *Present goal*: Describe the stable oscillations for the vacuum Bianchi type $\mathrm{VI}_{-1/9}$ models.

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- Fixed points: the *Robinson-Trautman (RT)*, the *plane wave arc ($\mathrm{PW}^{\pm}$)*, and the *Kasner circle*:

$$\mathrm{K}^{\bigcirc} := \left\{ (\Sigma_1, \Sigma_2, \Sigma_3, 0, 0, 0, 0) \in \mathbb{R}^7 \;\middle|\; \begin{array}{r} 1 - \Sigma^2 = 0, \\ \Sigma_1 + \Sigma_2 + \Sigma_3 = 0 \end{array} \right\}.$$

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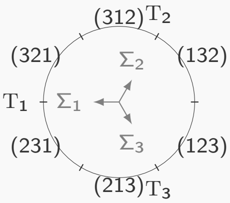

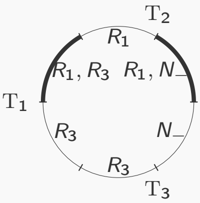

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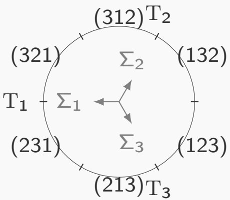
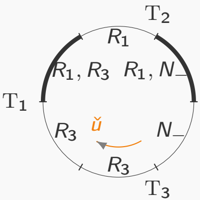
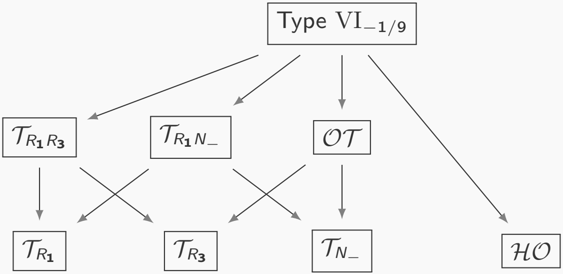




- We parametrize points in (132) by the Kasner parameter $u \in [1, \infty]$, which has 6 representations $\check{u}$, one in each sector.

Stratification of higher dimensional invariant sets



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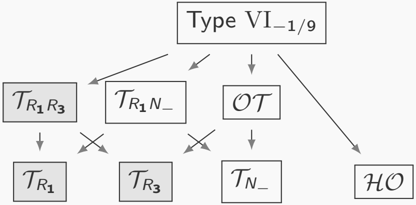

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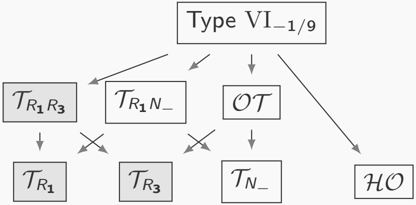

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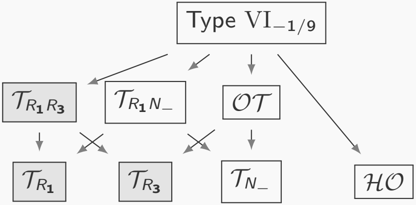

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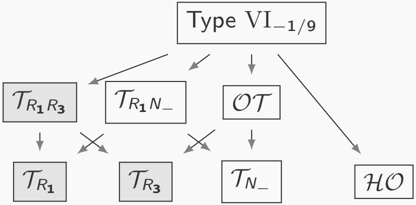

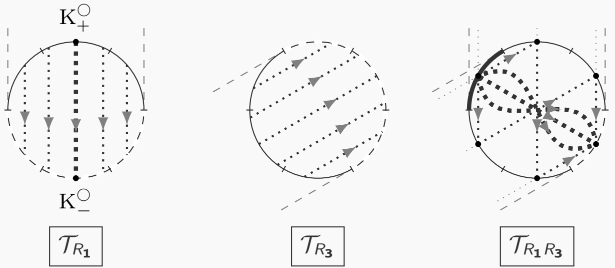

Figure: Projection of heteroclinic orbits onto in $(\Sigma_1, \Sigma_2, \Sigma_3)$-space.

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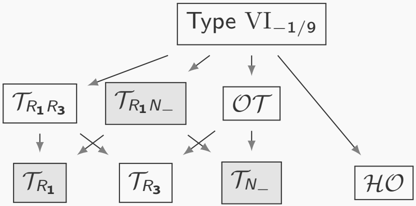

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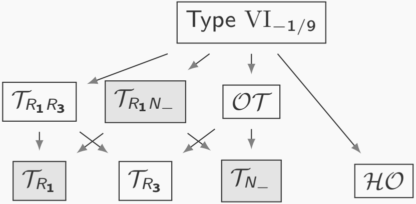

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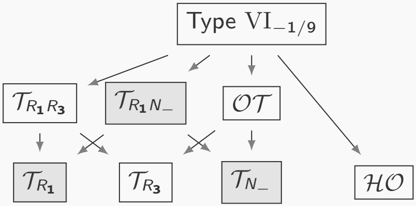

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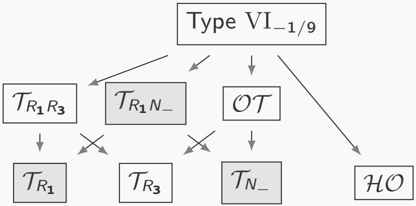

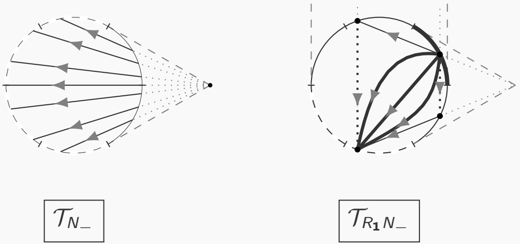

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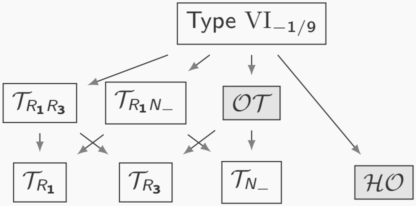

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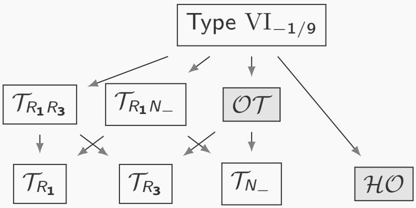

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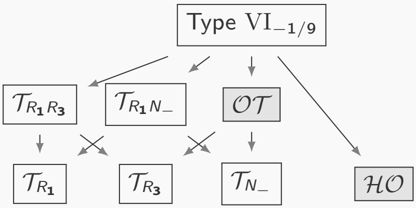

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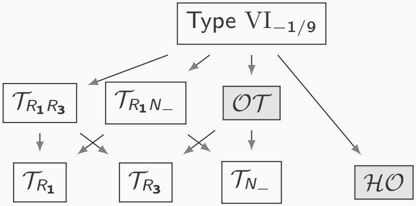

Figure: Schematic structure of the sets $\mathcal{OT}$ (left) and $\mathcal{HO}$ (right).

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- *Next goal:* construct examples of heteroclinic chains under the BKL map, which are candidates to yield stable oscillations.

Discrete dynamics

Hexagon description for each Kasner parameter u

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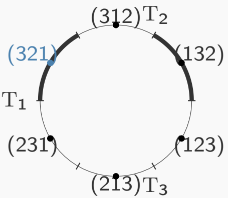

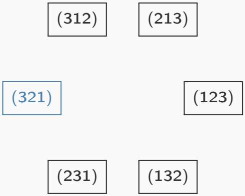

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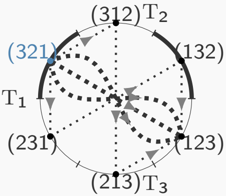

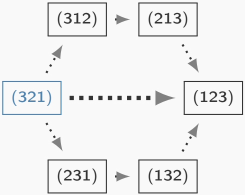

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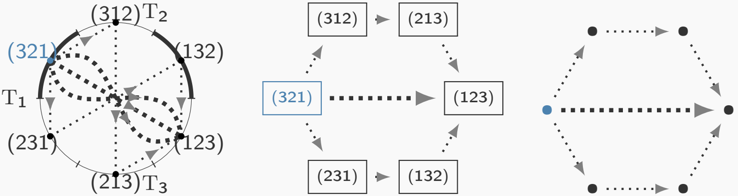
- The sectors notation are omitted and replaced with dots.



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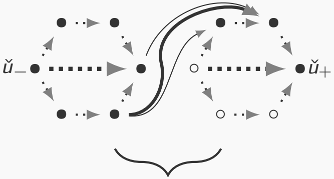
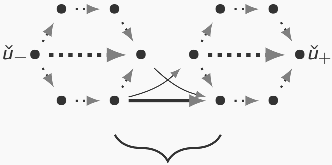


No era change: $u_+ = u_- - 1$.



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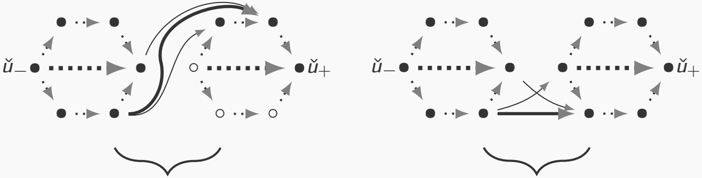

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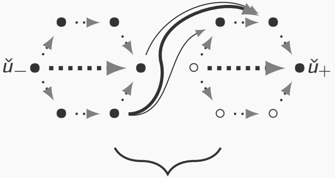

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- *All heteroclinic chains can be represented using hexagons and we can describe all their isolated structures!*

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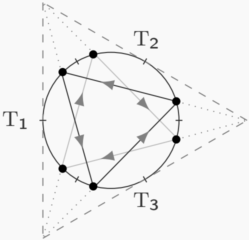

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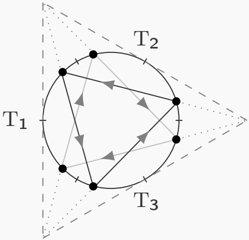

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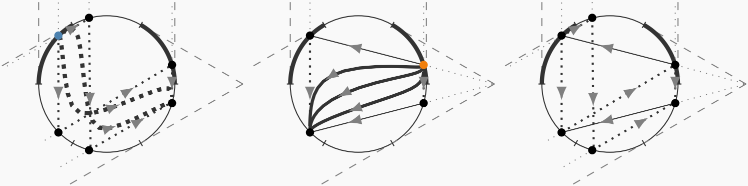

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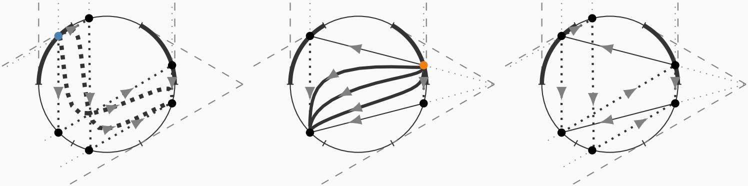

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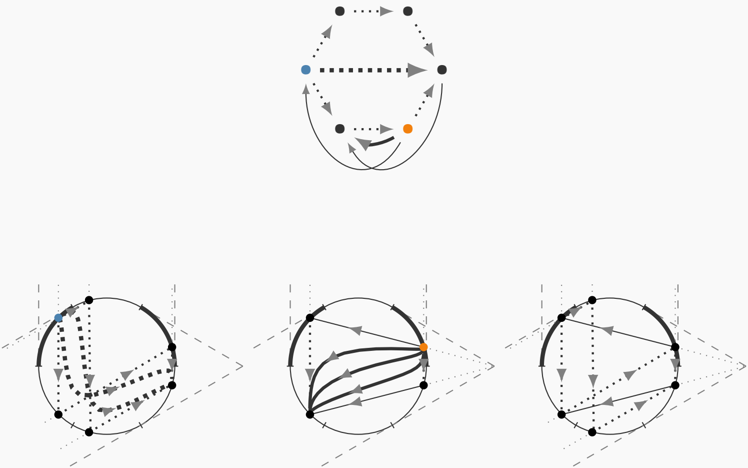

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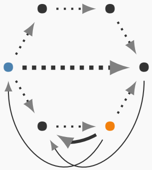

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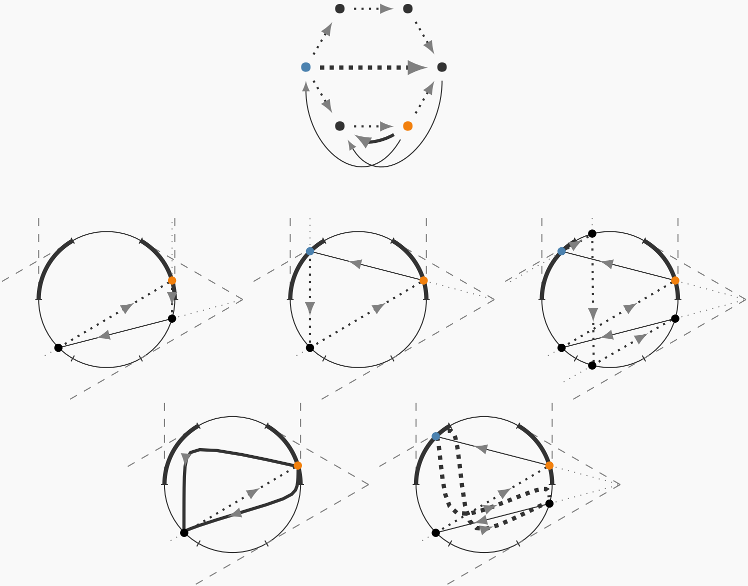

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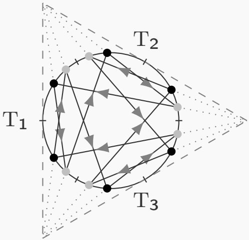

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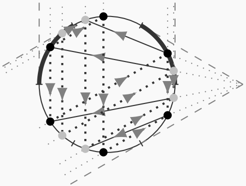

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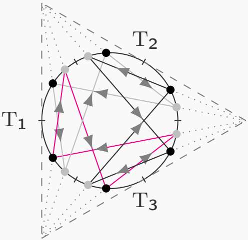

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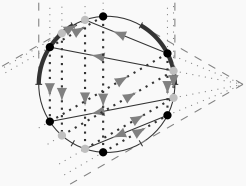
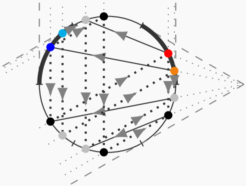


Figure: Type $\mathrm{VI}_{-1/9}$ (we omit the multiple-transitions).

Example: the silver ratio $u = 1 + \sqrt{2}$



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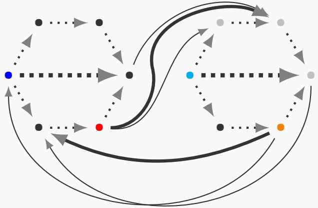

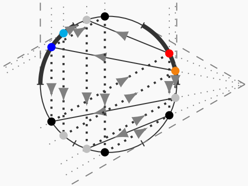

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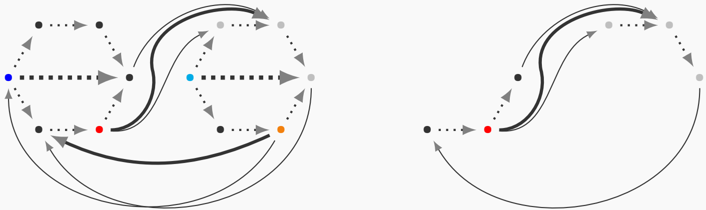

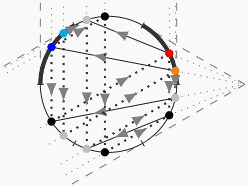

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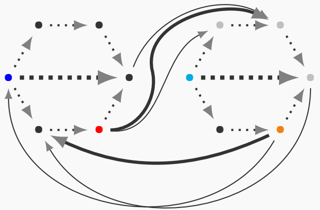

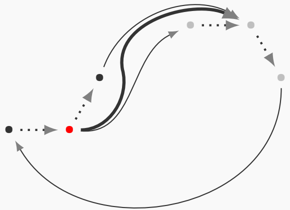

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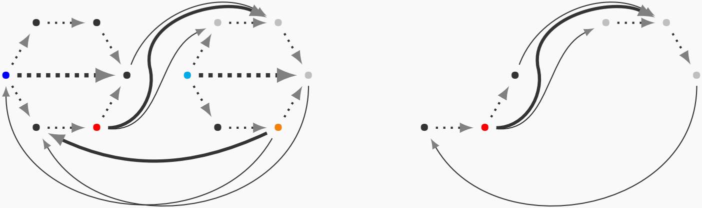

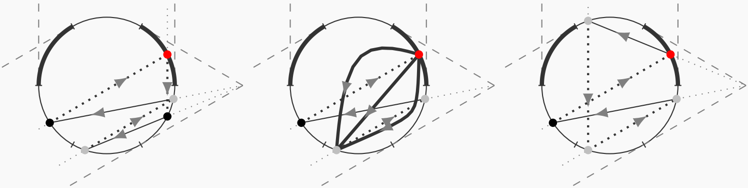

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Thank you.