Gravitational collapse to extremal Reissner–Nordström and the third law of black hole thermodynamics

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joint work with Ryan Unger (Stanford)

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

Black hole thermodynamics is a proposed close mathematical **analogy** between black hole dynamics and classical thermodynamics.

Black hole thermodynamics

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Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon

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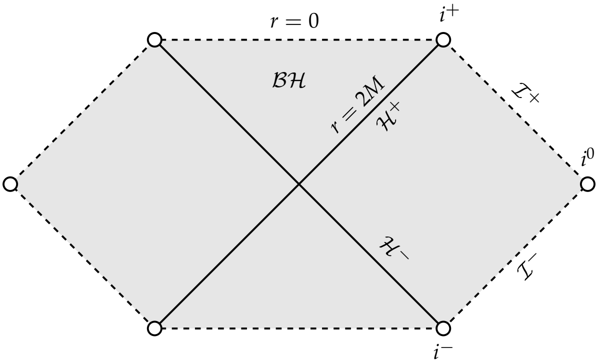
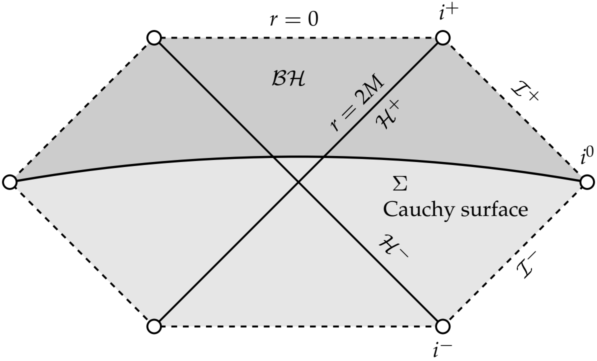
- Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

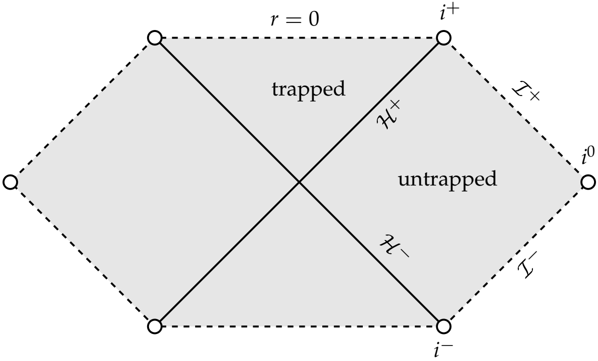
Refresher on Schwarzschild



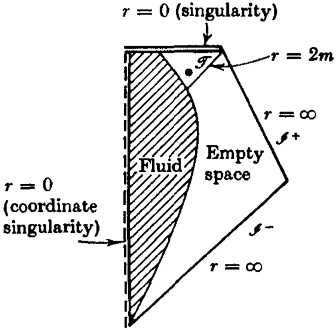


Maximally extended Schwarzschild is the unique maximal Cauchy development of the data induced on a spacelike hypersurface $\Sigma \cong \mathbb{R} \times S^2$ as depicted here.

Refresher on Schwarzschild

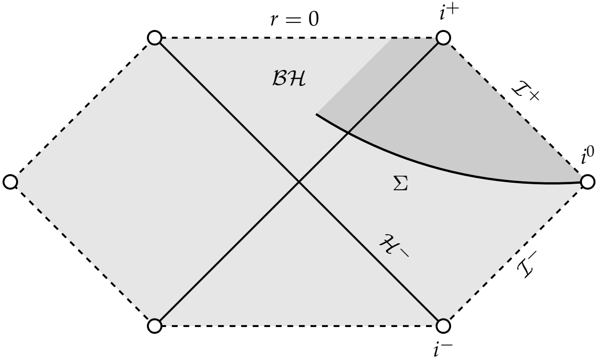


The black hole interior is foliated by **trapped spheres** (both future null expansions negative).

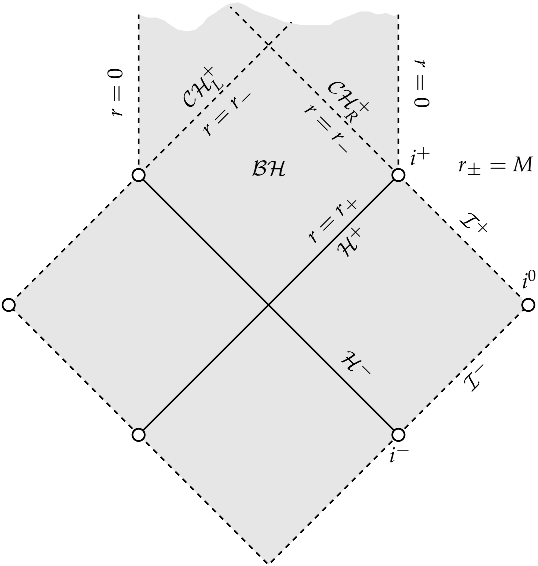


Penrose diagram of gravitational collapse. One-ended Cauchy data!

Refresher on gravitational collapse



Refresher on subextremal Reissner–Nordström: $0 < |e| < M$



$$r_\pm = M \pm \sqrt{M^2 - e^2}$$

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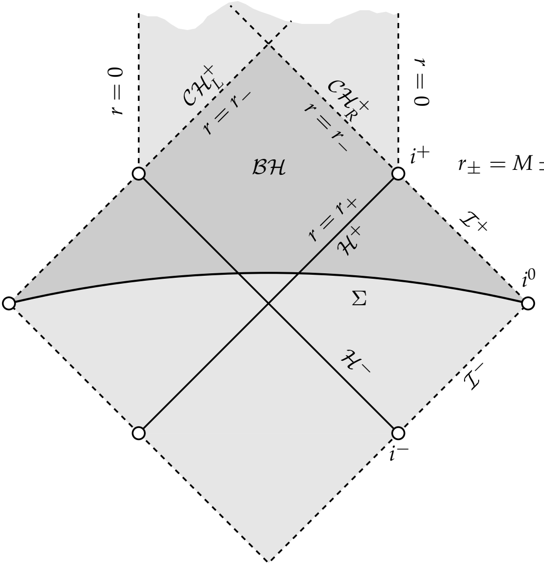

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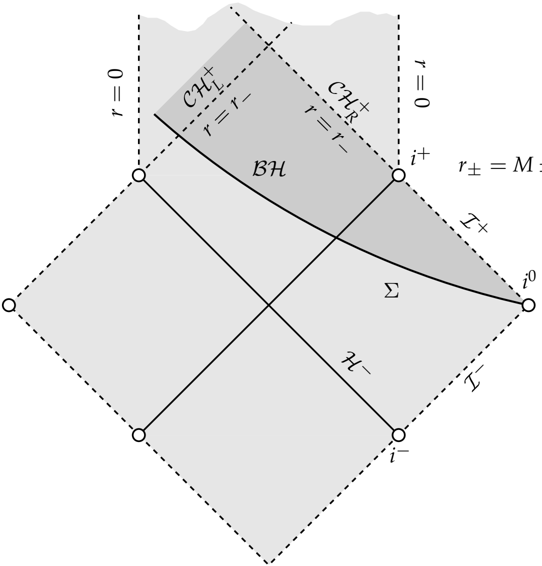

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Non-negative Hawking mass $m \doteq \frac{r}{2}(1 - g(\nabla r, \nabla r))$ requires $r \geq \frac{e^2}{2M}$.

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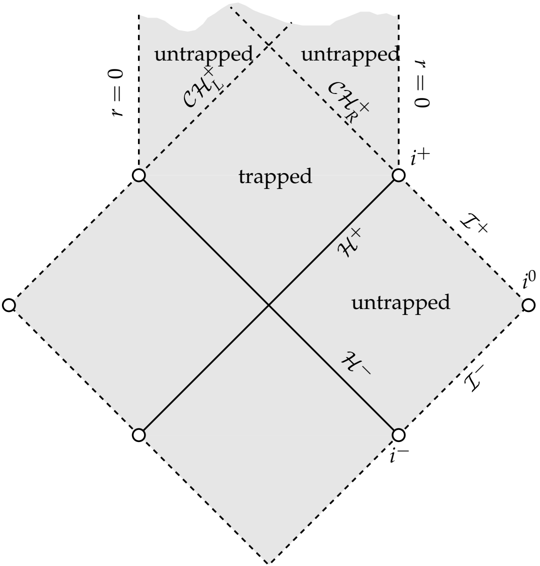

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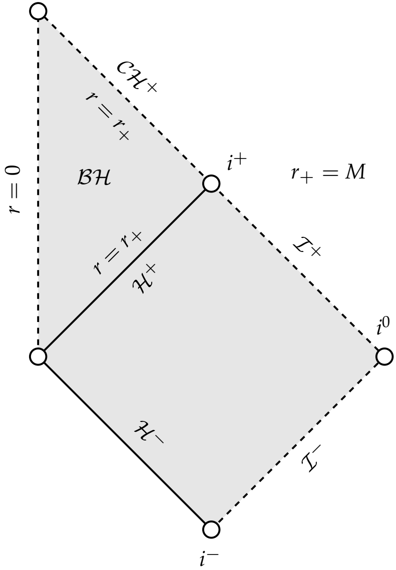

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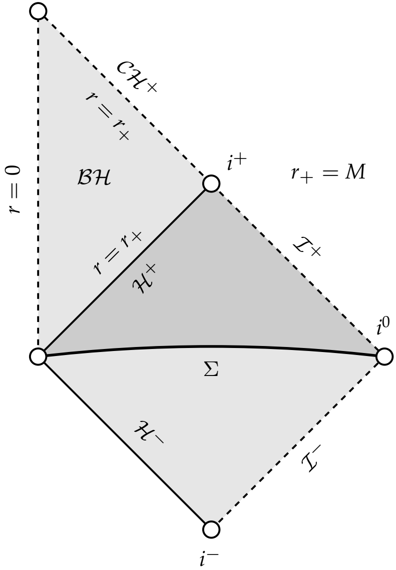

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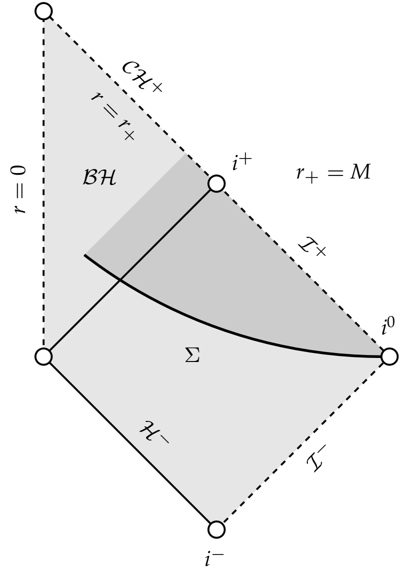

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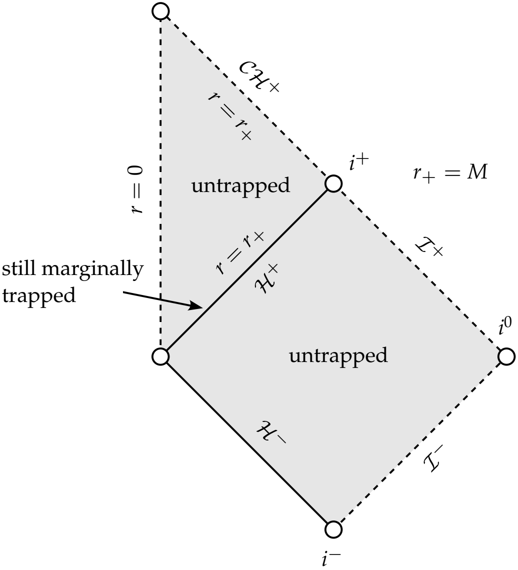
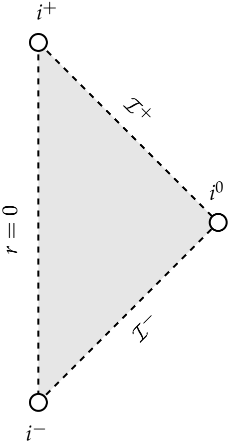


Refresher on superextremal Reissner–Nordström: $0 < M < |e|$



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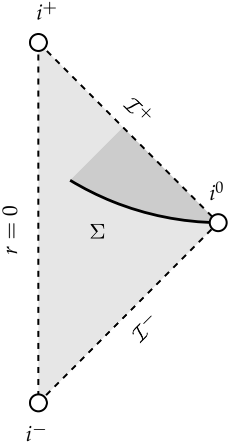

Non-negative Hawking mass requires $r \geq \frac{e^2}{2M}$.

Surface gravity κ of Reissner–Nordström

- RN with mass M and charge e, $|e| \leq M$, has

$$\kappa = 2\pi T = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

- **Subextremal:** $\kappa > 0$
- **Extremal:** $\kappa = 0$

Original formulation of Bardeen–Carter–Hawking:

The Third Law

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W. Israel[a]
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ISRAEL'S FORMULATION OF THE THIRD LAW

> A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

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4. Weak energy condition must be enforced.
   - Otherwise: counterexample using **charged null dust**. [Sullivan–Israel '80]

Conjecture (The third law, BCH '73, Israel '86).

A subextremal black hole cannot become extremal in finite time by any continuous process, ***no matter how idealized****, in which the spacetime and matter fields remain regular and obey the weak energy condition.*

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Theorem (K–Unger '22).

*There exists a precisely defined process in which a **subextremal** black hole becomes **extremal** in **finite time**, evolving from **regular** initial data in the Einstein–Maxwell charged scalar field system.*

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Theorem (K–Unger '22).

*There exists a precisely defined process in which a **subextremal** black hole becomes **extremal** in **finite time**, evolving from **regular** initial data in the Einstein–Maxwell charged scalar field system. In particular, the "third law of black hole thermodynamics" is **false**.*

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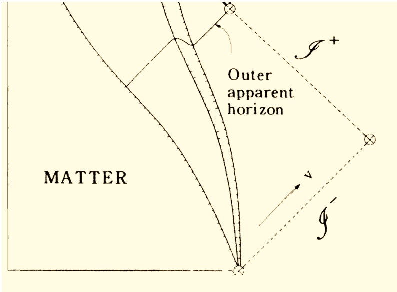
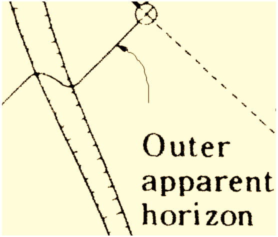


Israel argues by contradiction. Assume:

- First incoming matter flux creates (dynamical) subextremal apparent horizon.
- Second matter flux pushes the horizon to become to extremal.

ISRAEL'S ARGUMENT II



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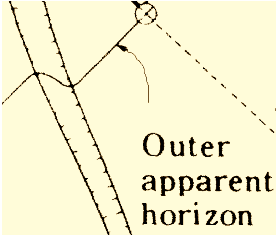

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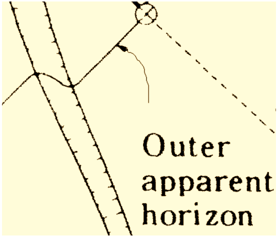

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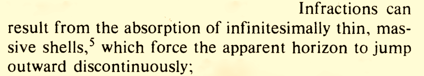

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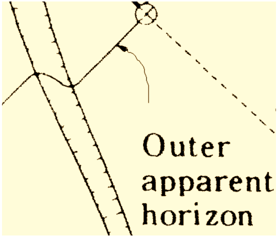

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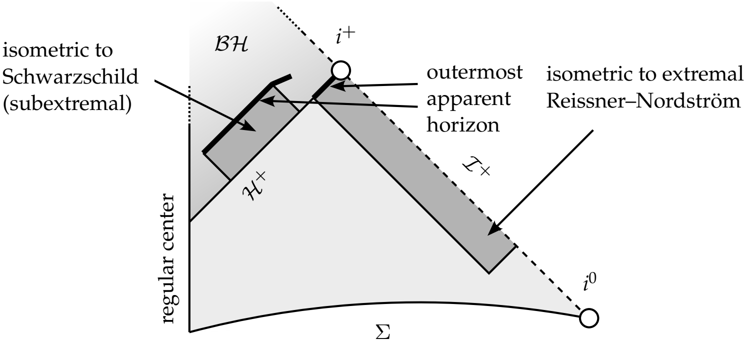
However, outer apparent horizon **can jump in smooth spacetimes.**

COUNTEREXAMPLE TO THE THIRD LAW



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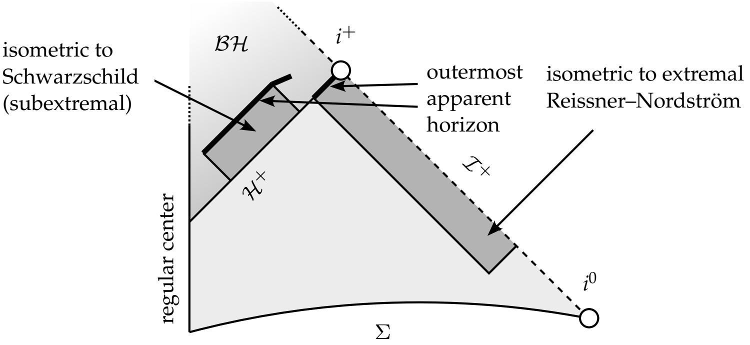

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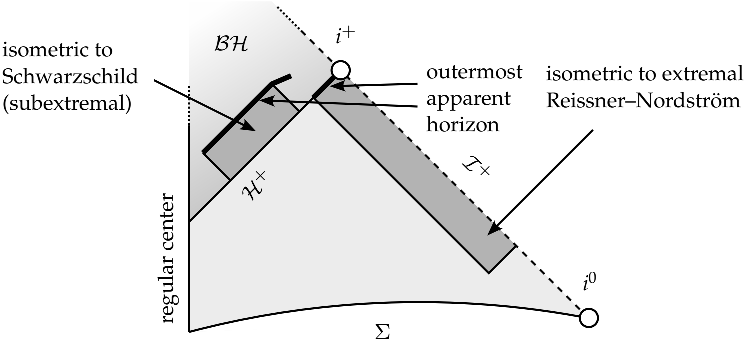

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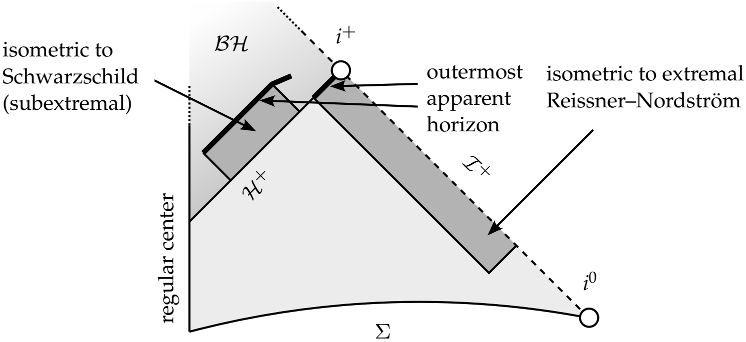

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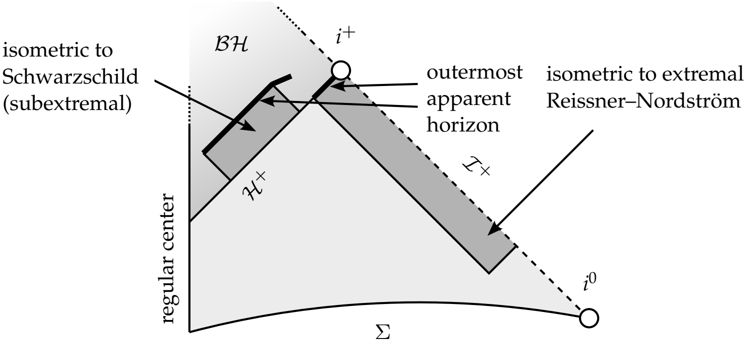

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- Dominant energy condition ($\Rightarrow$ weak energy condition)

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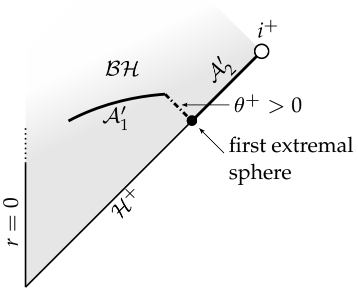

Outermost apparent horizon becomes disconnected the instant the black hole becomes extremal!

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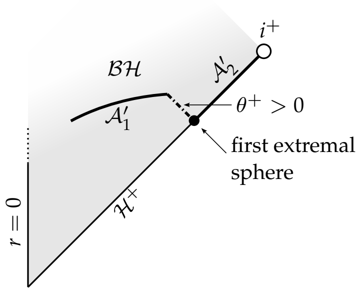

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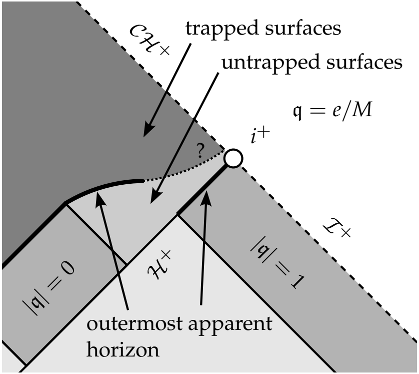
This is a feature, not a glitch!

Interior structure of third law violating solutions



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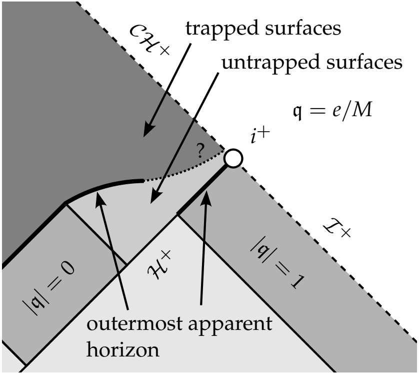

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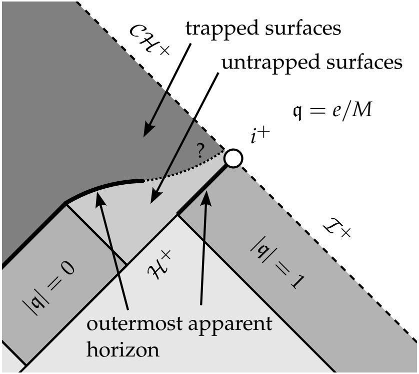
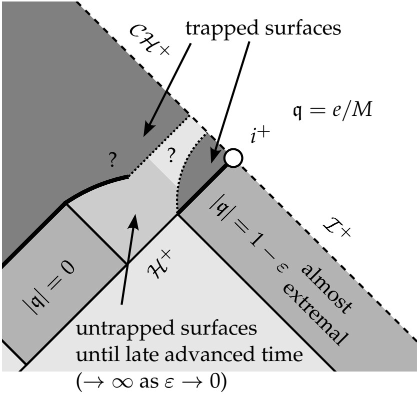


- The outermost apparent horizon becomes disconnected, yet the spacetime is regular.
- Trapped surfaces persist for all time.

So where do the trapped surfaces go?

The geometry of a $|\mathfrak{q}| = 1 - \varepsilon$ example converges to a $|\mathfrak{q}| = 1$ example as $\varepsilon \to 0$.



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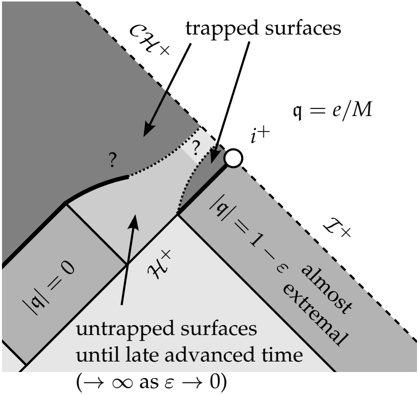

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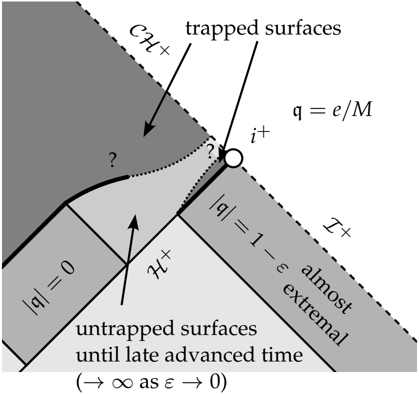

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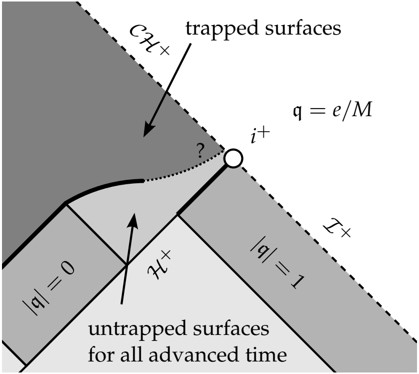

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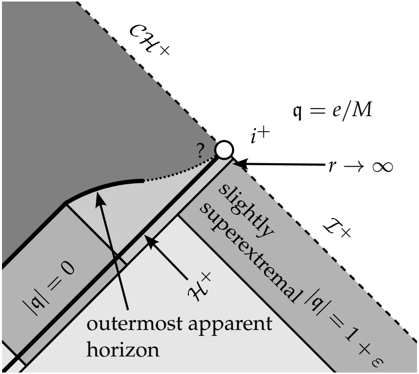

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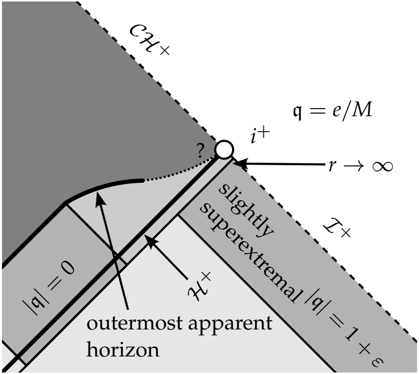

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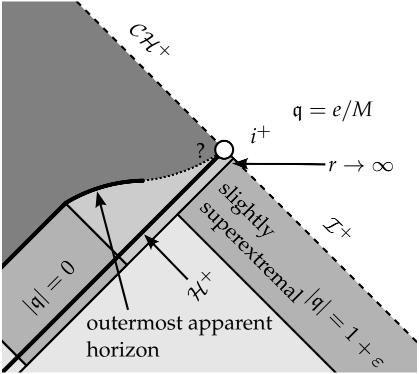

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The event horizon jumping associated to extremal horizons and the stability of this local critical behavior was conjectured by [DAFERMOS–HOLZEGEL–RODNIANSKI–TAYLOR '21].

Aside: Overcharging

Bardeen–Carter–Hawking:

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- Overcharging has been definitively **disproved** in sph. symmetry [KOMMEMI '13].
- Part of the spacetime is isometric to superextremal Reissner–Nordström $\not\Rightarrow$ there exists a naked singularity!

Einstein–Maxwell-charged scalar field system

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\begin{aligned}
R_{\mu\nu}(g) - \tfrac{1}{2}R(g)g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + T^{\mathrm{CSF}}_{\mu\nu}\right)\\
\nabla^{\mu}F_{\mu\nu} &= 2\mathfrak{e}\,\mathrm{Im}(\phi\overline{D_\nu\phi})\\
g^{\mu\nu}D_\mu D_\nu\phi &= 0\\
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Toy model: Einstein-scalar field in spherical symmetry

- $\mathcal{M}^{3+1} = \mathcal{Q}^{1+1} \times S^2$

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- Raychaudhuri's equations (constraints)

$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_u \phi)^2$$
$$\partial_v \left(\frac{\partial_v r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_v \phi)^2$$

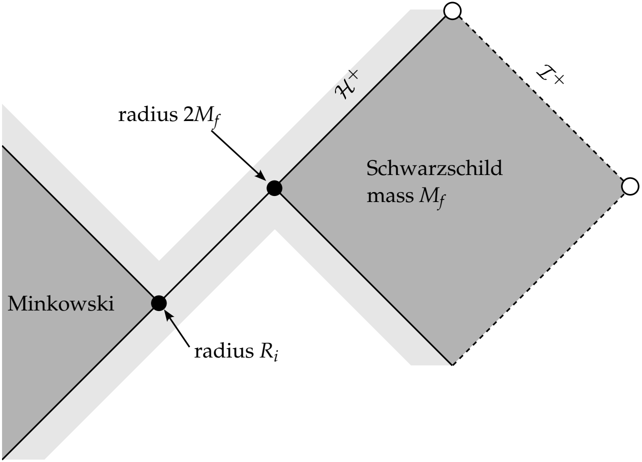
Hawking mass $m \doteq \frac{r}{2}(1 + 4\Omega^{-2}\partial_v r \partial_u r)$:

$$\partial_v m = 2r^2 \Omega^{-2} (-\partial_u r)(\partial_v \phi)^2$$

MINKOWSKI TO SCHWARZSCHILD GLUING

In our disproof we use a technique to construct solutions called **characteristic gluing**.

See [ARETAKIS–CZIMEK–RODNIANSKI, CHRUŚCIEL–CONG] for Einstein vacuum equations



Goal:

Set up characteristic data such that radii and Hawking masses have **a priori specified values** and $\phi, \partial_v^j\phi, \partial_u^j\phi$.

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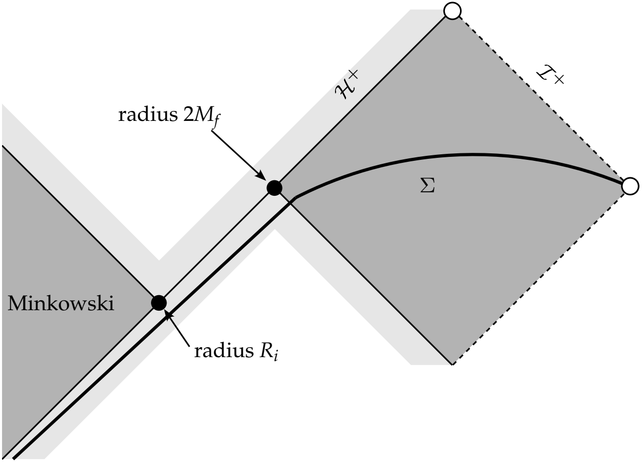

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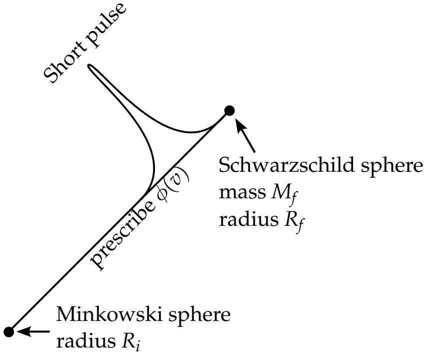
Theorem (K.–Unger '22).
For any $k \in \mathbb{N}$, $M_f > 0$ and $0 < R_i < R_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild sphere with radius R_f and mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.

A first approach and the issue of transverse derivatives

- On $v \in [0,1]$ use gauge $\Omega^2 = 1$ we impose $-\partial_u r(1) \gg 1 \Rightarrow |\partial_v r|, \Delta r \ll 1$ (short pulse [Christodoulou])
- Intermediate value thm: $\exists$ amplitude of ϕ such that $M_f = \int_0^1 2r^2(-\partial_u r)(\partial_v \phi)^2 dv$



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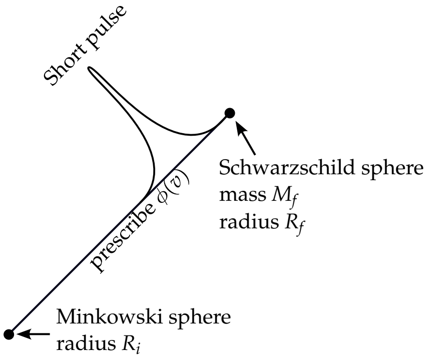
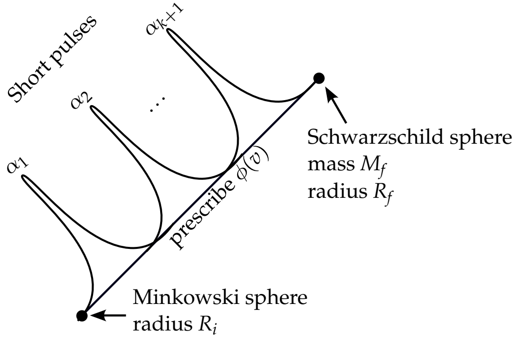


This is not enough because:

- Transverse derivative $\partial_u \phi$ is transported and sourced by ϕ along outgoing cone: $\partial_v(\partial_u \phi) = -\partial_u \phi \partial_v \log r - \partial_v \phi \partial_u \log r$.
- Generic choice of profile can only satisfy **either** $\partial_u \phi(0) = 0$ **or** $\partial_u \phi(1) = 0$.
- However, **gluing requires both** and also higher transverse derivatives.

Idea of the proof: Schwarzschild



- Scalar field ansatz $\phi_\alpha(v) = \sum_{1\leq j\leq k+1} \alpha_j \chi_j(v), \quad \alpha \in \mathbb{R}^{k+1}$

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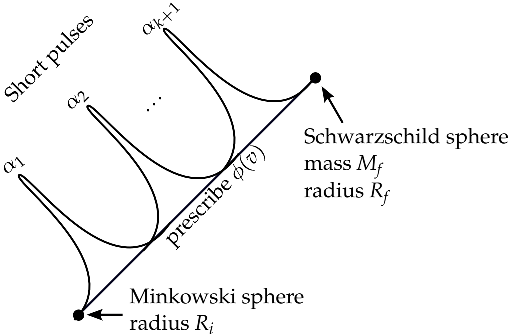

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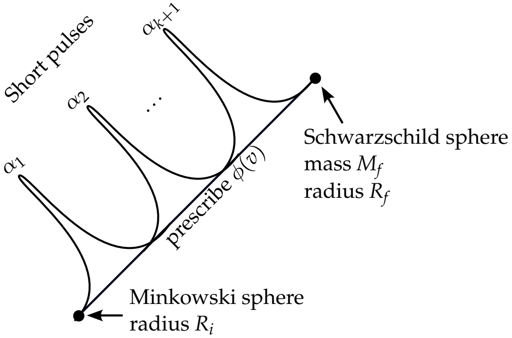

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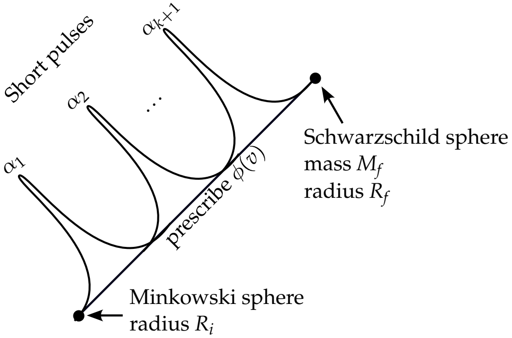

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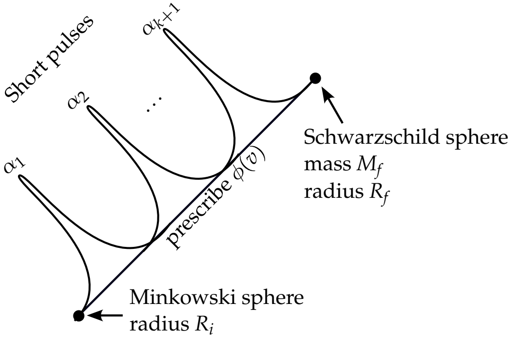

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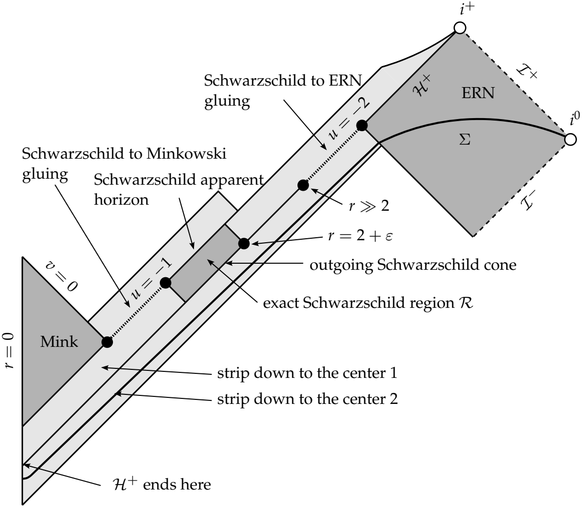
- **Borsuk–Ulam theorem:** there exists α_* such that

$$\left(\partial_u \phi_{\alpha_*}(1), \ldots, \partial_u^k \phi_{\alpha_*}(1)\right) = 0.$$

DISPROOF OF THE THIRD LAW

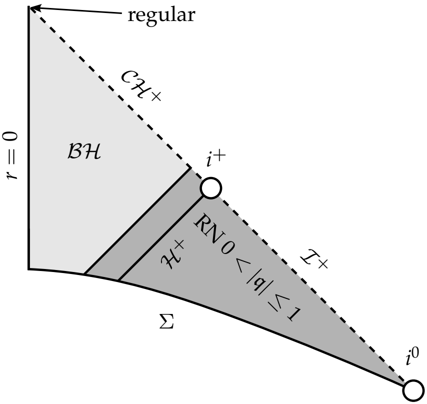


Poincaré inequality obstruction: $\partial_v m \sim (-\partial_u r) r^2 (\partial_v \phi)^2$ but $\partial_v Q \sim r^2 \phi \partial_v \phi$
$\Rightarrow$ A short pulse **cannot** produce an extremal black hole.

Beyond the disproof of the third law,
the gluing method allows us to construct further interesting behavior.

Theorem (K.–Unger '22).

*There exist black holes **without** trapped surfaces.*



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Penrose's theorem **does not** guarantee the stability of their black hole-ness.

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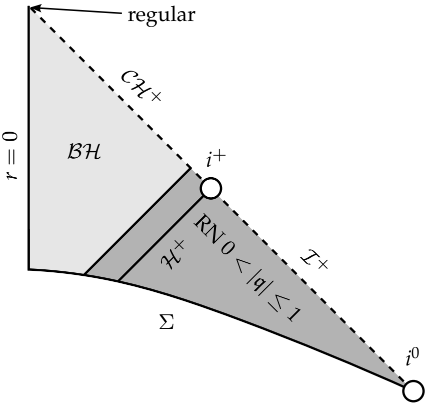

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Such black holes could be natural candidates for **critical solutions**!

Living Rev. Relativity, **10**, (2007), 5
http://www.livingreviews.org/lrr-2007-5
(Update of lrr-1999-4)

LIVING REVIEWS *in relativity*

Critical Phenomena in Gravitational Collapse

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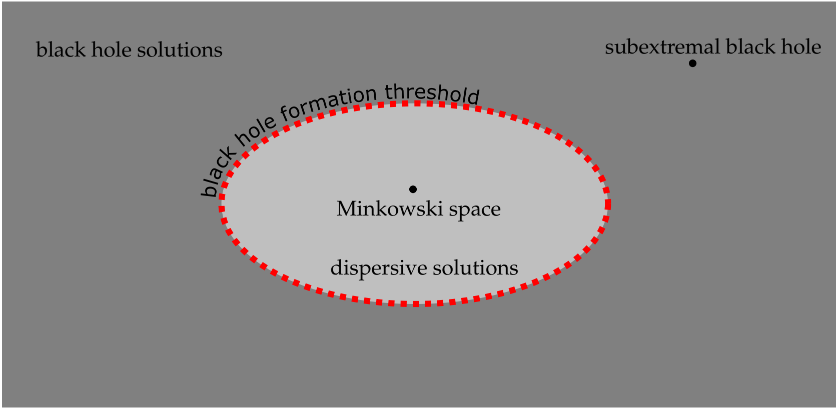
Cartoon picture of moduli space



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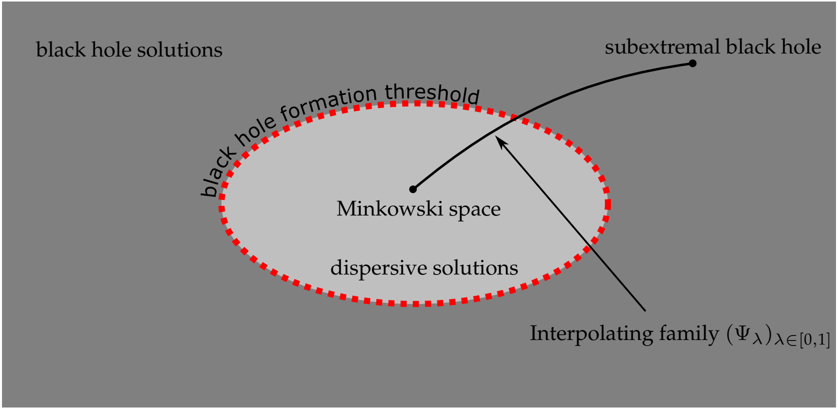

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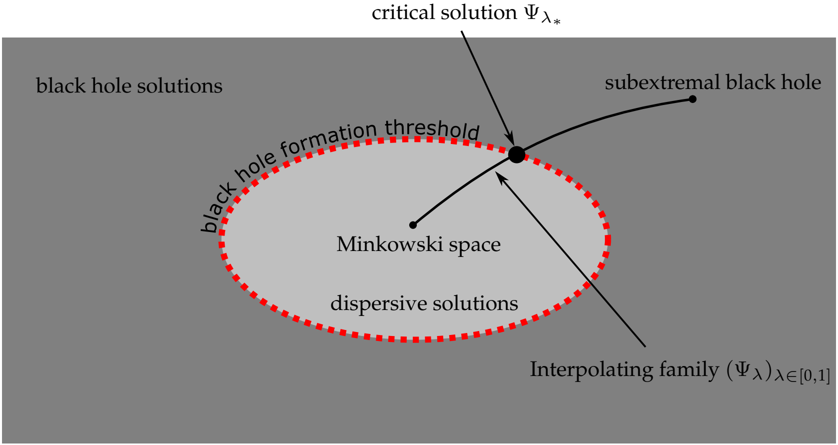

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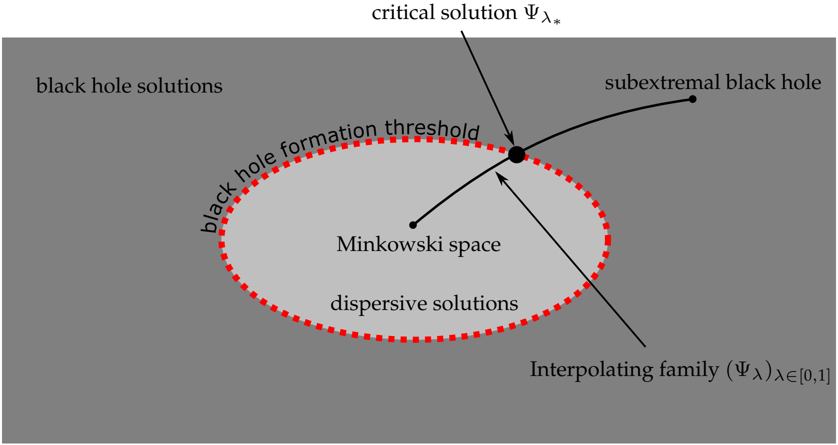

Numerics for sph. symm. Einstein-scalar field: Ψ_{λ_*} leads to a **naked singularity**
[CHOPTUIK '93, ...]

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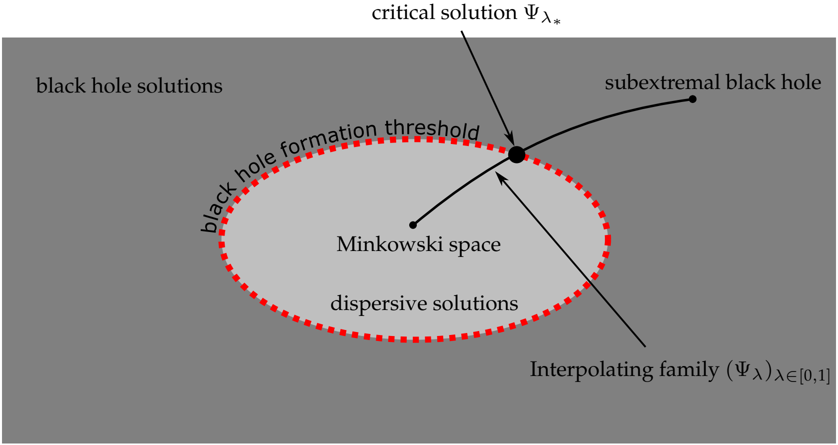

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Also numerics suggesting **star-like objects** as Ψ_{λ_*} for Einstein–Klein–Gordon/Vlasov
[BRADY, CHAMBERS, GONCALVES, REIN, RENDALL, SCHAEFFER, ...]

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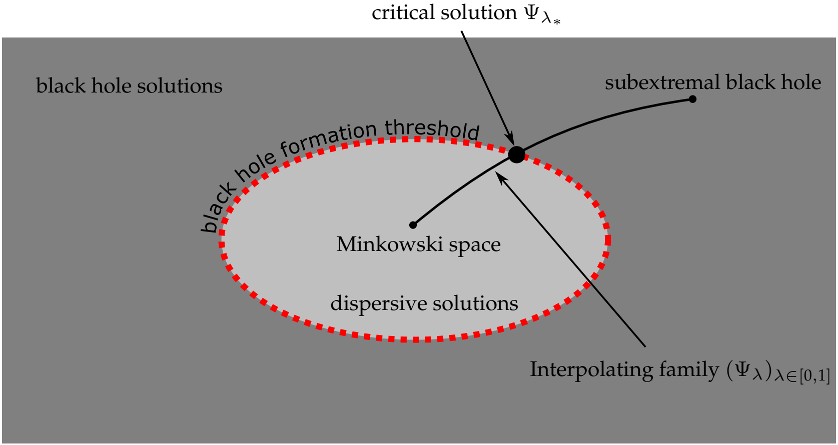

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It is an open problem to make any of these numerics rigorous!

THEOREM: EXTREMAL CRITICAL COLLAPSE

We consider **self-gravitating charged plasma**: Einstein–Maxwell–Vlasov system

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\begin{cases}
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*There exist **extremal** black holes on the black hole formation threshold!*

We consider **self-gravitating charged plasma**: Einstein–Maxwell–Vlasov system

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} + \int_{P_x^{\mathfrak{m}}} p_\mu p_\nu f \, d\mu_x^{\mathfrak{m}}\right), \\ \nabla^\alpha F_{\mu\alpha} = \mathfrak{e} \int_{P_x^{\mathfrak{m}}} p_\mu f \, d\mu_x^{\mathfrak{m}}, \\ p^\mu \frac{\partial}{\partial x^\mu} f - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} f = -\mathfrak{e} F^\mu{}_\alpha p^\alpha \frac{\partial}{\partial p^\mu} f. \end{cases}$$

Theorem (K.–Unger '24).

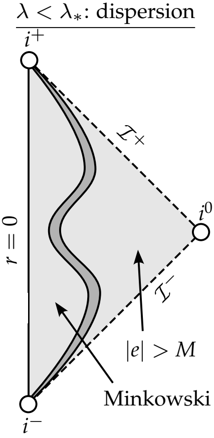
There exists a smooth 1*-parameter family of solutions* $\{D_\lambda\}_{\lambda\in[0,1]}$ *and a critical value* $\lambda_* \in (0,1)$ *such that:*

- *If* $0 \le \lambda < \lambda_*$*, the solution* ***disperses*** *to Minkowski space and* ***no*** *black hole forms.*
- *If* $\lambda = \lambda_*$*, an* ***extremal*** *black hole forms.*
- *If* $\lambda_* < \lambda \le 1$*, a* ***subextremal*** *black hole forms.*

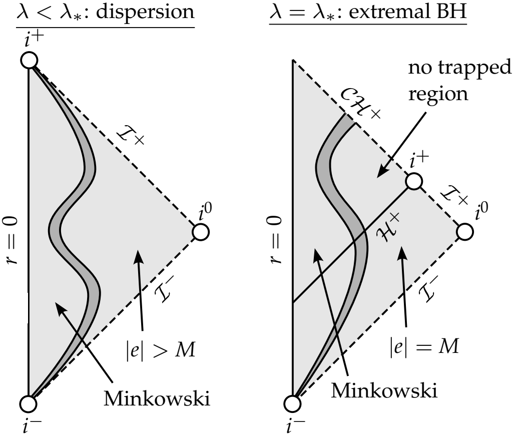
There exist ***extremal*** *black holes on the black hole formation threshold!*

We call this phenomenon **Extremal Critical Collapse**.

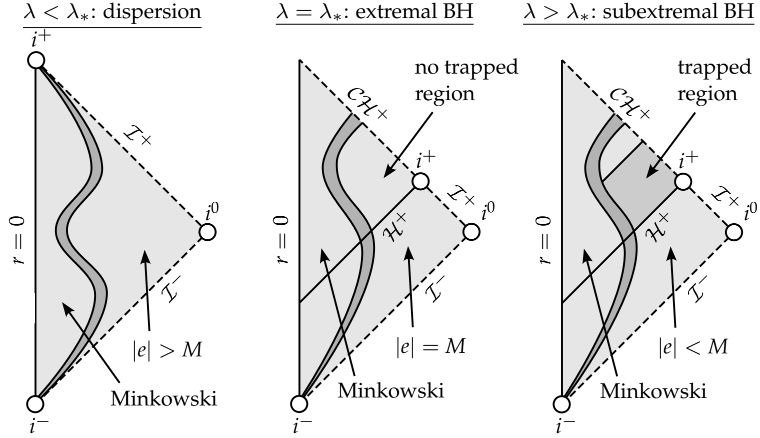
Penrose diagram: Extremal critical collapse



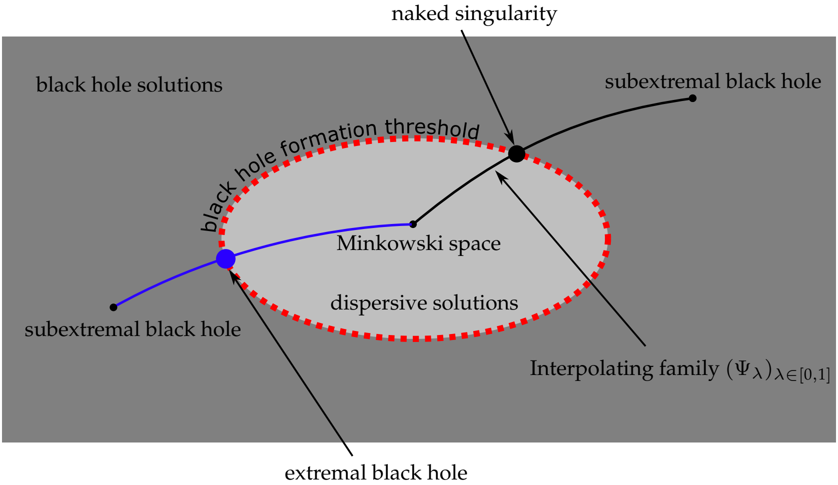
PENROSE DIAGRAM: EXTREMAL CRITICAL COLLAPSE



Penrose diagram: Extremal critical collapse



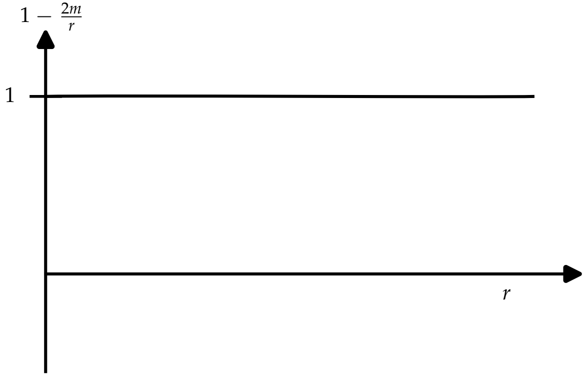
Cartoon picture of moduli space



EXTREMAL CRITICAL COLLAPSE: $1-\frac{2m}{r}$ ALONG LATE INGOING CONE

In spherical symmetry: **trapped** sphere if and only if $1-\frac{2m}{r}<0$.

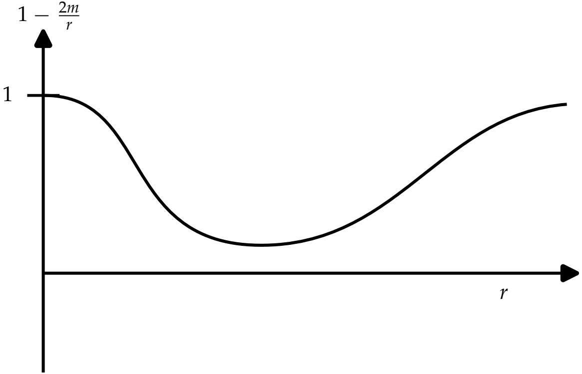
<u>$\lambda = 0$: Minkowski</u>



Extremal critical collapse: $1 - \frac{2m}{r}$ along late ingoing cone

In spherical symmetry: **trapped** sphere if and only if $1 - \frac{2m}{r} < 0$.

$\lambda < \lambda_*$: dispersion



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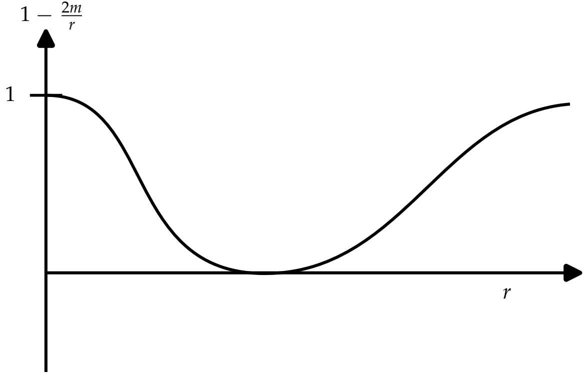
$\lambda = \lambda_*$: extremal black hole



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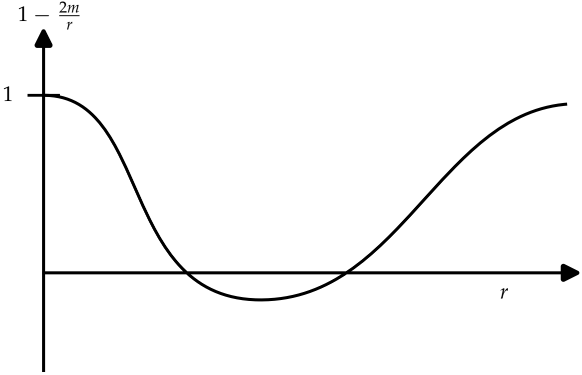
$\lambda > \lambda_*$: subextremal black hole



Aspects about the proof

Consider a singular toy model: Einstein–Maxwell–charged null dust

$$
\begin{aligned}
R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + T_{\mu\nu}\right), \\
\nabla^{\alpha} F_{\mu\alpha} &= \mathfrak{e}\rho k_{\mu}, \\
k^{\nu}\nabla_{\nu} k^{\mu} &= \mathfrak{e} F^{\mu}{}_{\nu} k^{\nu}, \\
\nabla_{\mu}(\rho k^{\mu}) &= 0,
\end{aligned}
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The system is **not well-posed** but an explicit, **singular** solution can be written down in terms of the ingoing charged Vaidya solution (Ori '91) and "free" functions ϖ_{in}, Q_{in} with $\dot{\varpi}_{in} \geq 0$ and $\dot{Q}_{in} \geq 0$ for $D(V,r) \doteq 1 - \frac{2\varpi_{in}(V)}{r} + \frac{Q_{in}^2(V)}{r^2}$:

$$g_{\mathrm{in}}[\varpi_{\mathrm{in}}, Q_{\mathrm{in}}] \doteq -D(V,r)\, dV^2 + 2\, dV\, dr + r^2 \gamma,$$

$$F \doteq -\frac{Q_{\mathrm{in}}}{r^2}\, dV \wedge dr,$$

$$k \doteq \frac{\mathfrak{e}}{\dot{Q}_{\mathrm{in}}} \left(\dot{\varpi}_{\mathrm{in}} - \frac{Q_{\mathrm{in}} \dot{Q}_{\mathrm{in}}}{r} \right) (-\partial_r), \quad \rho \doteq \frac{(\dot{Q}_{\mathrm{in}})^2}{\mathfrak{e}^2 r^2} \left(\dot{\varpi}_{\mathrm{in}} - \frac{Q_{\mathrm{in}} \dot{Q}_{\mathrm{in}}}{r} \right)^{-1}$$

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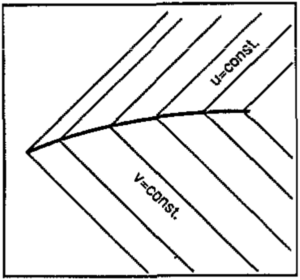
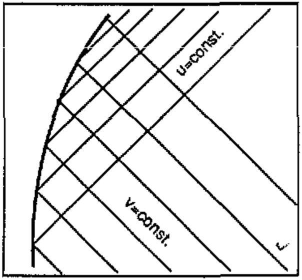
$$\text{Bounce radius: } r_b \doteq \frac{Q_{in}\dot{Q}_{in}}{\dot{\varpi}_{in}}$$

Note: $T_{\mu\nu} = \rho k_{\mu} k_{\nu}$ violates null energy condition if $r < r_b$.

Ori's interpretation: Once an ingoing fluid trajectory hits the bounce hypersurface $\Sigma_b = \{r = r_b\}$, it has to change direction from **ingoing** to **outgoing**.

Spacelike bounce hypersurface





$\Sigma_b := \{r = r_b\}$ is **spacelike** $\Rightarrow$ Explicit surgery with an outgoing Vaidya solution is possible such that second fundamental form is continuous. (Ori '91)

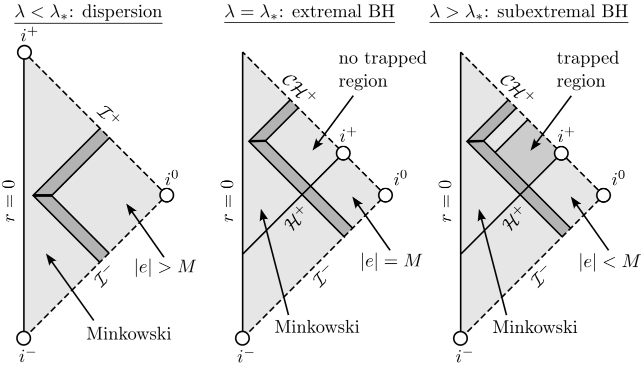
However, solution is still singular across Σ_b:

$$\rho \notin L^\infty, \quad N := \rho k \notin C^0$$

$\Sigma_b := \{r = r_b\}$ being **spacelike** is a **teleological** assumption!

Extremal Critical Collapse in null dust model

Theorem (K.–Unger '24).
The charged null dust model exhibits extremal critical collapse.



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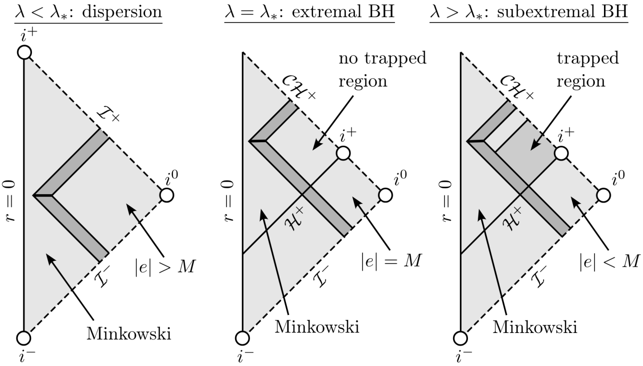
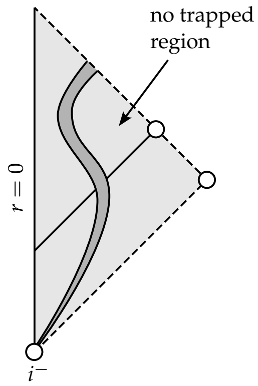
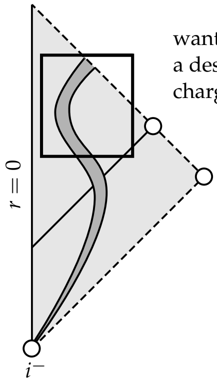


Proof idea: Instead of prescribing free function ϖ, Q as in Ori's model, we **directly** prescribe the geometry of Σ_b: find solutions to a system of ODEs and differential inequalities.

Smooth Extremal Critical Collapse: Vlasov case

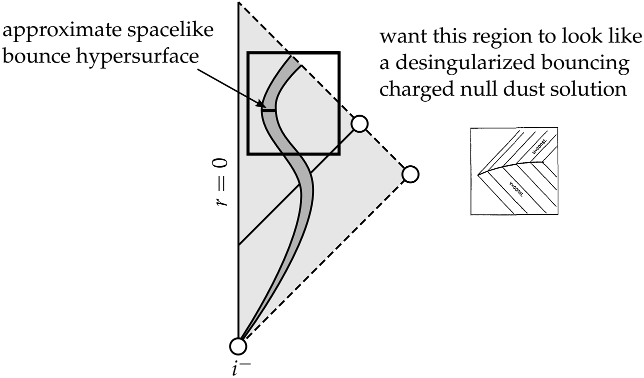


CONSTRUCTION OF BOUNCING CHARGED VLASOV BEAMS



want this region to look like a desingularized bouncing charged null dust solution

Construction of bouncing charged Vlasov beams



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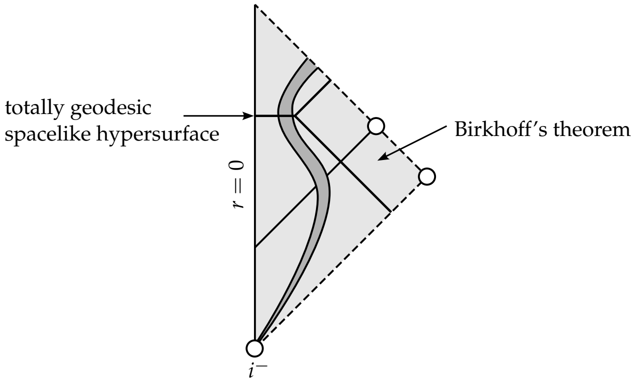

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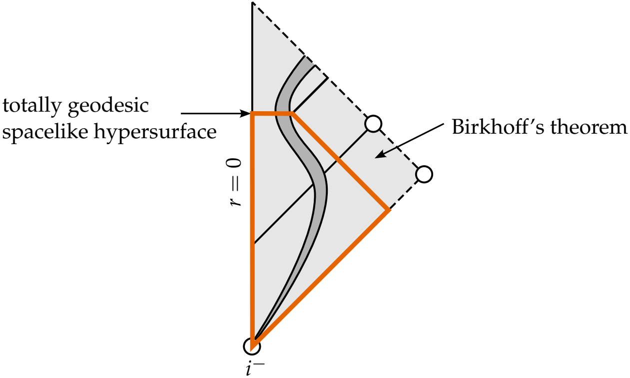

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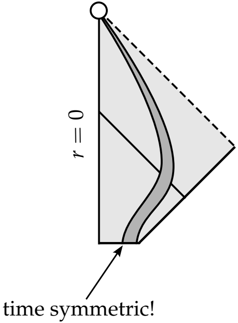
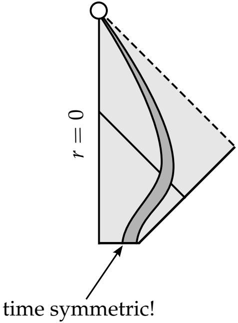


Construction of bouncing charged Vlasov beams

particles have conserved
angular momentum $\ell^2 = r^2 \not{g}(p, p)$



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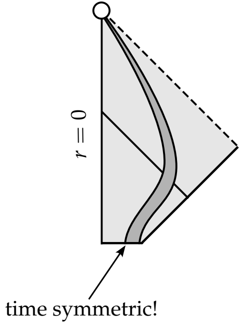

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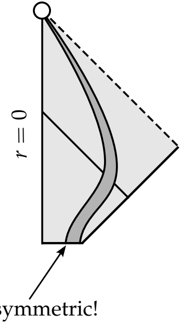
charged null dust:
monokinetic Maxwell–Vlasov with $\ell = 0$

Construction of bouncing charged Vlasov beams



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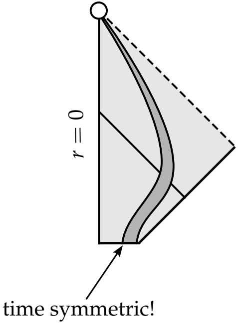

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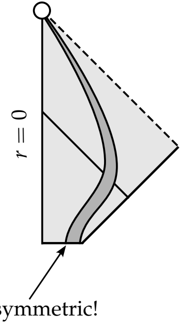
to behave like dust

Construction of bouncing charged Vlasov beams



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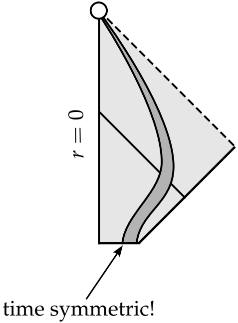
$|J^{\mathrm{EM}}| \gtrsim 1 \Longrightarrow f_0 \approx \varepsilon^{-3}$ ($f \rightharpoonup \delta'(p)$ as $\varepsilon \to 0$)

Construction of bouncing charged Vlasov beams



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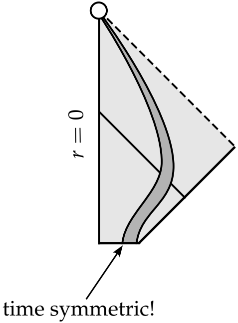
Construction of bouncing charged Vlasov beams



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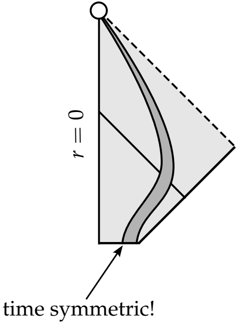

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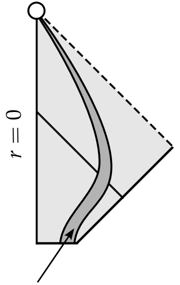
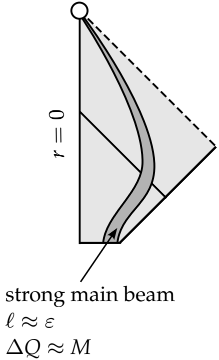
Difficulty: **Instability of em-geodesic flow** at the inner edge of the beam, where charge repulsion is arbitrarily small.


$r = 0$
strong main beam
$\ell \approx \varepsilon$
$\Delta Q \approx M$

Construction of bouncing charged Vlasov beams

the most important feature to resolve is the outward acceleration near the bounce hypersurface



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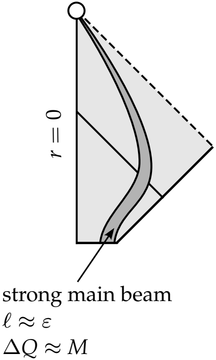

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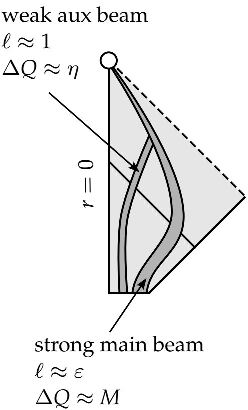
we employ a weak "auxiliary beam" to impart charge $0 < \varepsilon \ll \eta \ll 1 \implies$ **stabilizes** the main beam



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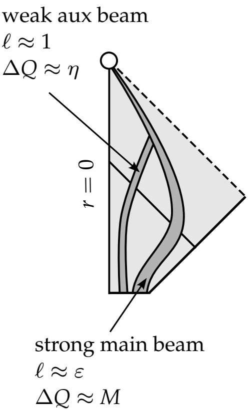
CONSTRUCTION OF BOUNCING CHARGED VLASOV BEAMS



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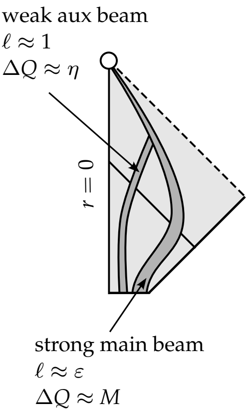

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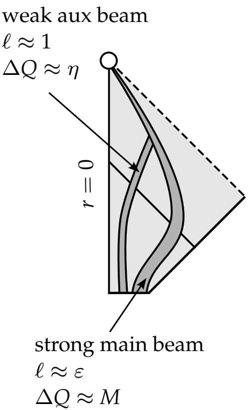

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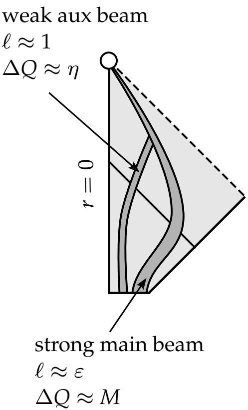
Construction of bouncing charged Vlasov beams



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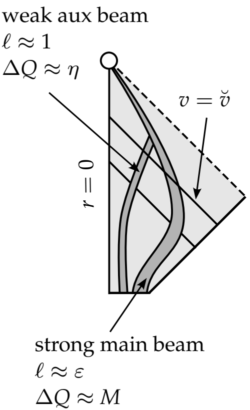

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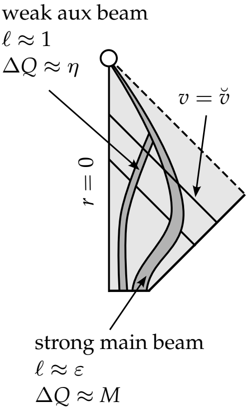

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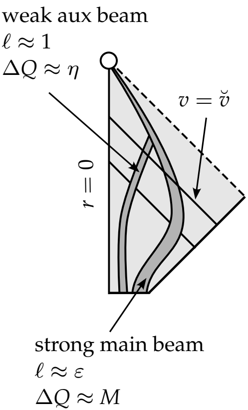
hierarchy of scales $0 < \mathfrak{m} \ll \varepsilon \ll \eta \ll \breve{v}^{-1} \ll 1$

Construction of bouncing charged Vlasov beams



the most important feature to resolve is the outward acceleration near the bounce hypersurface

we employ a weak "auxiliary beam" to impart charge $0 < \varepsilon \ll \eta \ll 1 \Longrightarrow$ **stabilizes** the main beam

the auxiliary beam bounces due to angular momentum repulsion, not charge

null structure: T^{uu}, T^{uv} better in ε than T^{vv}

monotonicity: $\partial_u Q \leq 0$, $\partial_v Q \geq 0$

dispersion proved using energy estimates at a late time $\breve{v} \gg 1$

hierarchy of scales $0 < \mathfrak{m} \ll \varepsilon \ll \eta \ll \breve{v}^{-1} \ll 1$

For Vlasov we make fundamental use of the repulsive effects of **charge** and **angular momentum**.

Conjecture.
*Extremal critical collapse is a **stable** phenomenon.*

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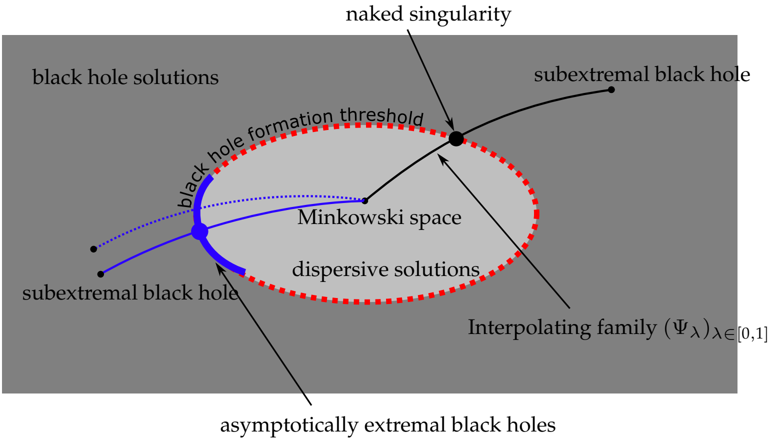
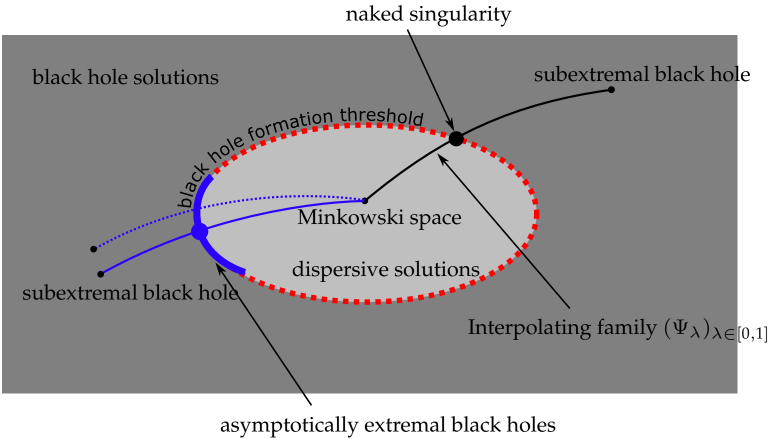


Conjecture.

Extremal critical collapse is a ***stable*** *phenomenon.*



- This is also a non-trivial statement about the **interiors** of black holes.

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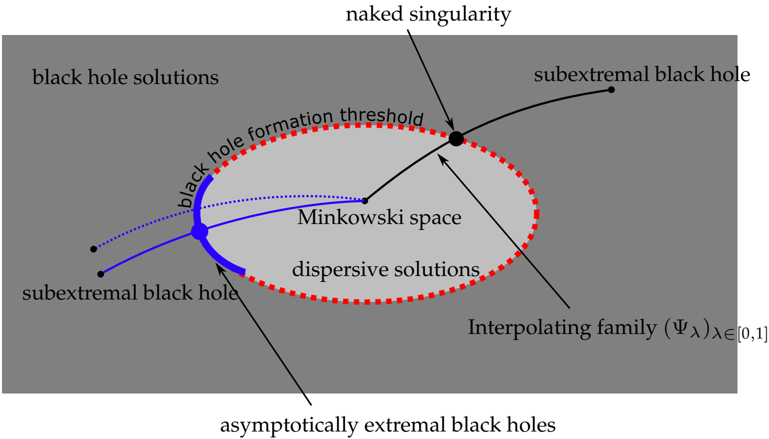

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- Further difficulty: **Aretakis instability** associated to extremal horizons

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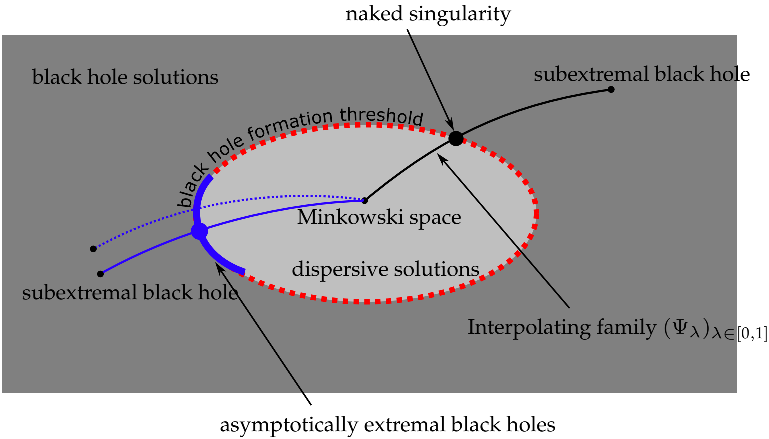

- This is also a non-trivial statement about the **interiors** of black holes.
- Further difficulty: **Aretakis instability** associated to extremal horizons
- **Theorem.** Extremal Reissner–Nordström is codimension 1 stable. [ANGELOPOULOS–K.–UNGER '24]

THE VACUUM CASE: THE THIRD LAW

Far less is known in **vacuum** and even the third law has not yet been disproved.

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Conjecture.
There exist Cauchy data for the Einstein vacuum equations

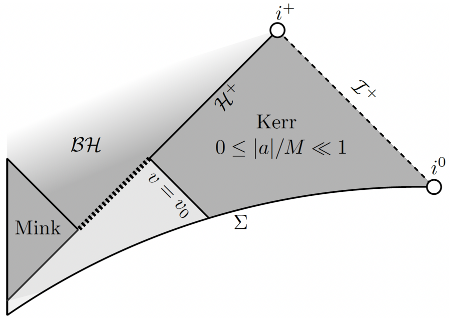
$$R_{\mu\nu} = 0$$

which undergo gravitational collapse and form an exactly Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Kerr event horizon at a later advanced time. In particular, ***already in vacuum, the "third law of black hole thermodynamics" is false.***

Theorem (K.–Unger, '23).
For any $0 \leq |a| \ll M$, there exist Cauchy data for the Einstein vacuum equations

$$R_{\mu\nu} = 0$$

which undergo gravitational collapse and form an ***exactly*** *Kerr event horizon at a finite advanced time with specific angular momentum a and mass M.*



In principle, however, *extremal critical collapse*, its *stability*, and the *revised picture of moduli space*

The vacuum case: Extremal critical collapse

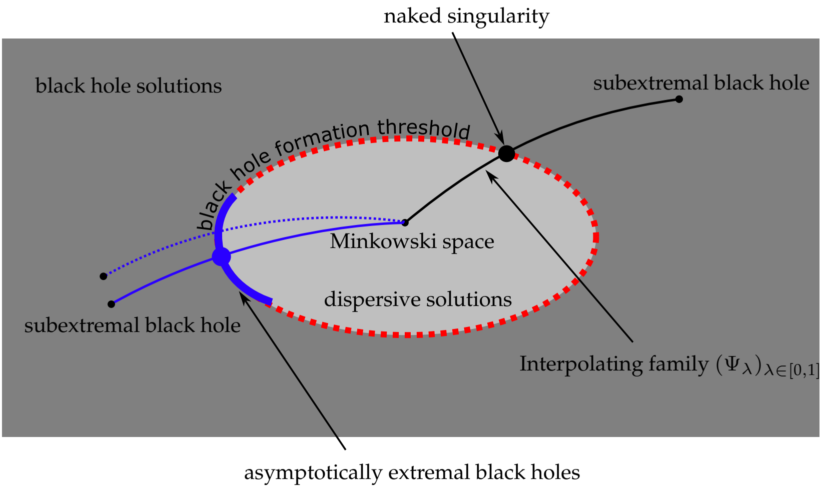
In principle, however, *extremal critical collapse*, its *stability*, and the *revised picture of moduli space* can be conjectured to also hold true in **vacuum**

The vacuum case: Extremal critical collapse

In principle, however, *extremal critical collapse*, its *stability*, and the *revised picture of moduli space* can be conjectured to also hold true in **vacuum** with extremal Reissner–Nordström replaced by **extremal Kerr**.

However, this is a very difficult open problem and also relates to understanding

- the codimension stability and stability of extremal and near-extremal black holes [Dafermos–Holzegel–Rodnianski–Taylor]
- the nonlinear ramifications of horizon instabilities associated to extremal Kerr [Aretakis, Gajic].
- See essay by M. Dafermos on: "The stability problem for extremal black holes."


naked singularity
black hole solutions
subextremal black hole
black hole formation threshold
Minkowski space
dispersive solutions
subextremal black hole
Interpolating family $(\Psi_\lambda)_{\lambda\in[0,1]}$
asymptotically extremal black holes

Thank you!