A Low-regularity Riemannian Positive Mass Theorem for Non-Spin Manifolds with Distributional Curvature

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Introduction

Riemannian positive mass theorem (PMT):

- Let M^n, $3 \leq n < 8$ be a smooth manifold. Let g be a complete asymptotically flat C^∞ metric. If $R[g] \geq 0$, then $m[g] \geq 0$ and $m[g] = 0$ iff (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, \delta)$.

$$m[g] = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \sum_{i,j=1}^{n} (\partial_i g_{ij} - \partial_j g_{ii}) \nu_j dS$$

- First proved by [SY79a] for $n < 8$.
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Why Low-regularity?

- "Synthetic" lower bounds scalar curvature (Alexandrov/RCD Spaces)
- Contains quite some interesting analysis: Regularisation of distributional curvature, Friedrichs-type lemma, conformal method, Ricci-flow, etc.

Introduction

Low-regularity PMT?

- $g \in W^{1,n}_{\text{loc}} \cap C^0$

$$Riem[g] \approx \underbrace{\partial\Gamma}_{\in \mathcal{D}'(M)} + \underbrace{\Gamma \cdot \Gamma}_{\in L^{n/2}_{\text{loc}}(M)}$$

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$$\langle R[g], \varphi\rangle \geq 0 \quad \forall\varphi \geq 0 \in \mathcal{D}(M)$$

- ADM-mass is not well defined since $\partial g \in L^n(M)$.

$$m[g] = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \sum_{i,j=1}^{n} (\partial_i g_{ij} - \partial_j g_{ii})\nu_j dS$$

(Usually, it is assumed that the metric is smooth away from a singular subset.)

Introduction

Low-regularity PMT Literature

Let Σ and K be a hypersurface and a compact set on M, respectively.

- $g \in C^{0,1}(M)$ and $g \in C^{\infty}(M \backslash \Sigma)$ [Mia02; MS12]
- $g \in W^{2,n/2}_{\text{loc}}(M)$ and $g \in C^{\infty}(M \backslash K)$ [GT14]
- $g \in C^{0}(M) \cap W^{1,n}_{\text{loc}}(M)$ (spin) [LL15]
- Many other contributions: [JSZ22; Lee12; ST02; Li20]...

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- Many other contributions: [JSZ22; Lee12; ST02; Li20]...

Present work

- $g \in C^{0}(M) \cap W^{1,n}_{\text{loc}}(M)$ and $g \in C^{\infty}(M \backslash K)$ (non-spin)

Low-regularity PMT: Sketch of the proof

Theorem 1.1 (Nonnegativity)

Let M^n, $3 \leq n < 8$ be a smooth manifold. Let g be a complete asymptotically flat $C^0 \cap W^{1,n}_{\mathrm{loc}}$ metric and $g \in C^\infty(M \setminus K)$. If $R[g] \geq 0$ in $\mathcal{D}'$ sense, then $m[g] \geq 0$.

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Sketch of the proof:

1. Find $\{g_\varepsilon\}_{\varepsilon>0}$ smooth such that $g_\varepsilon \to g$ in $W^{1,n}_{\text{loc}}$ and uniformly in compact sets and $g|_{M\backslash K_\varepsilon} = g_\varepsilon|_{M\backslash K_\varepsilon}$ for some compact set K_ε
2. ...
3. Find conformal factor u_ε s.t $R[\widetilde{g}_\varepsilon] \geq 0$, where $\widetilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{n-2}} g_\varepsilon$
4. $m[\widetilde{g}_\varepsilon] \geq 0$ (smooth PMT) and $m[\widetilde{g}_\varepsilon] \to m[g] \implies m[g] \geq 0$

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Main challenge:

Step (3): Solve a PDE that only has solution if $\|R[g_\varepsilon]_-\|_{L^{n/2}(M)}$ is sufficiently small. However, we only have $R[g_\varepsilon] \to R[g]$ in $\mathcal{D}'$

Low-regularity PMT: Step 1 - Regularisation

Chartwise regularisation of metric by convolution on a compact set K

$$g_\varepsilon := g \star_M \rho_\varepsilon := \eta_{K^c} g + \sum_{i=1}^{m} \chi_i (\psi_i)_*^{-1} \left[\big((\psi_i)_* (\eta_i g) \big) * \rho_\varepsilon \right]$$

Lemma 1.1 ([GT14])

There exist smooth metrics g_ε and a compact set $K_\varepsilon \subset M$ with

1. $g_\varepsilon \to g$ *in* $W^{1,n}_{loc}(M)$ *and locally uniformly as* $\varepsilon \to 0$
2. $g_\varepsilon \equiv g$ *on* $M \backslash K_\varepsilon$

In particular, K_ε *is the closure of the* ε*-neighborhood of* K

Low-regularity PMT: Step 2 - Scalar curvature

- Problem: $R[g_\varepsilon] \to R[g]$ only in $\mathcal{D}'$
  $\rightsquigarrow R[g] \geq 0$ in $\mathcal{D}'$ doesn't help to control $\|R[g_\varepsilon]_-\|_{L^{\frac{n}{2}}(M)}$

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- Solution: compatibility of distinct regularisations

$$R[g_\varepsilon] - \underbrace{R[g] \star_M \rho_\varepsilon}_{\geq 0} \to 0 \text{ in } L^{n/2}_{\mathrm{loc}} \quad (g \in W^{1,n}_{\mathrm{loc}} \cap C^0)$$

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Key Idea: Friedrichs-type Lemma

In coordinates we denote $\mathfrak{g}_\varepsilon := (\psi_* g_\varepsilon)$, the relevant terms are

$$\underbrace{\mathfrak{g}_\varepsilon^{ij}\mathfrak{g}_\varepsilon^{ks}}_{=:a_\varepsilon}\underbrace{(\partial_s \mathfrak{g}_{lm})}_{=:f} * \rho_\varepsilon - (\underbrace{\mathfrak{g}^{ij}\mathfrak{g}^{ks}}_{=:a}\underbrace{\partial_s \mathfrak{g}_{lm}}_{=:f}) * \rho_\varepsilon \to 0 \text{ in } W^{1,\frac{n}{2}}_{\text{loc}}$$

$$\text{Prove } \boxed{a_\varepsilon f_\varepsilon - (af)_\varepsilon \to 0 \text{ in } W^{1,\frac{n}{2}}_{\text{loc}}}$$

$a \in W^{1,n}_{\text{loc}}$, $f \in L^n_{\text{loc}}$ and $C^\infty \ni a_\varepsilon \to a$ in $W^{1,n}_{\text{loc}}$, $f_\varepsilon := f * \rho_\varepsilon$

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- If $R[g] \geq 0$ distributionally and $g \in C^\infty(M \backslash K)$, then

$$\|R[g_\varepsilon]_-\|_{L^{n/2}(M)} \leq 2\|R[g_\varepsilon] - R[g] \star_M \rho_\varepsilon\|_{L^{n/2}(K_\varepsilon)} \xrightarrow{\varepsilon \to 0} 0$$

Low-regularity PMT: Step 3 - Conformal Method

- If $\|R[g_\varepsilon]_-\|_{L^{n/2}(M)}$ is sufficiently small, then the following system

$$\begin{cases} c_n \Delta_{g_\varepsilon} u_\varepsilon + R[g_\varepsilon]_- u_\varepsilon = 0 (*) \\ \lim_{x \to \infty} u_\varepsilon = 1 \end{cases}$$

has a C^2 positive solution u_ε on M [SY79b; Mia02].

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- Then, $\widetilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{n-2}} g_\varepsilon$ has scalar curvature given by

$$R[\widetilde{g}_\varepsilon] = u_\varepsilon^{-\frac{n+2}{n-2}} (-c_n \Delta_{g_\varepsilon} u_\varepsilon + R[g_\varepsilon] u_\varepsilon)$$
$$\overset{(*)}{=} u_\varepsilon^{-\frac{n+2}{n-2}} \left(R[g_\varepsilon]_+ u_\varepsilon \right) \geq 0$$

Low-regularity PMT: Step 4 - Convergence

- $R[\widetilde{g}_\varepsilon] \geq 0$ and smooth PMT $\implies m[\widetilde{g}_\varepsilon] \geq 0$
- The masses of g_ε and $\widetilde{g}_\varepsilon$ are related by

$$m[\widetilde{g}_\varepsilon] = m[g_\varepsilon] + 2A_\varepsilon$$

$$(2-n)\omega_{n-1}A_\varepsilon = \int_M |\nabla^{g_\varepsilon} u_\varepsilon|^2 - \frac{1}{c_n} R[g_\varepsilon]_- u_\varepsilon^2 d\mu_{g_\varepsilon}$$

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References

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