Reconstructing the Past with the Cosmological Time Function

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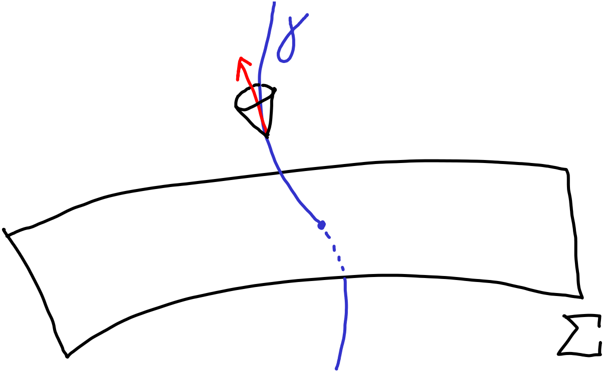
Cauchy time functions

- A subset $\Sigma \subset M$ in a spacetime (M, g) is called a Cauchy hypersurface if every inextendible causal curve intersects Σ exactly once.



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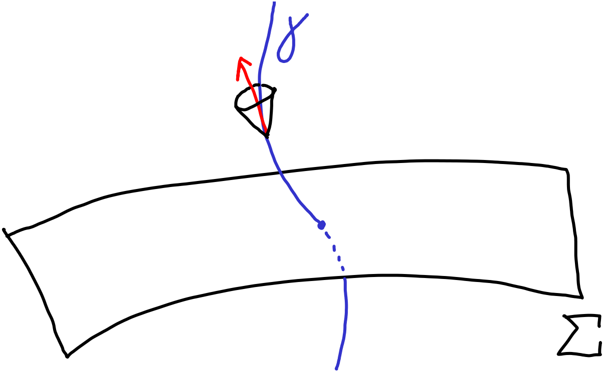

Used to specify initial data for Einstein equations

Cauchy time functions

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Theorem (Geroch 1970, Bernal–Sánchez 2005)

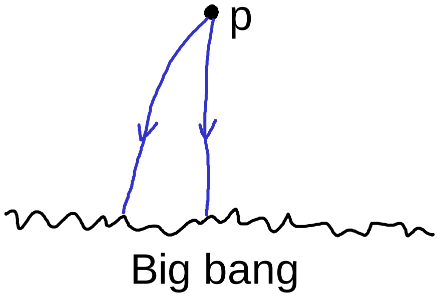
(M, g) is globally hyperbolic iff there exists a time function $\tau \colon M \to \mathbb{R}$ such that $\Sigma_t := \{\tau = t\}$ is a Cauchy hypersurface for every $t \in \mathbb{R}$.

Cosmological time function

$$\tau(p) := \sup\{L_g(\gamma) \mid \gamma \text{ past-directed causal starting at } p\}$$

where $L_g(\gamma)$ is the length (=proper time). Call τ regular if:

1. $\tau(p) < \infty$ for all $p \in M$
2. $\tau \circ \gamma \to 0$ along every past-inextendible causal curve γ



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Theorem (Andersson–Galloway–Howard 1999)

If τ is regular cosmological time function then:

1. τ is indeed a time function
2. (M, g) is globally hyperbolic
3. For every p, there exists a past-directed geodesic γ starting at p with $L_g(\gamma) = \tau(p)$

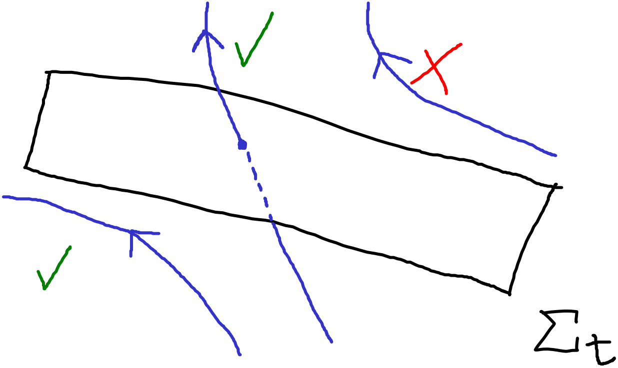
$$\begin{aligned} \tau \text{ regular cosmological time} \quad &\Longrightarrow \quad (M, g) \text{ globally hyperbolic} \\ &\Longrightarrow \quad \text{there is a Cauchy time function} \end{aligned}$$

But is τ itself a Cauchy time function?

τ regular cosmological time $\implies$ (M, g) globally hyperbolic
$\implies$ there is a Cauchy time function

But is τ itself a Cauchy time function? No $\ddot{\frown}$

In general, Σ_t is only a future Cauchy hypersurface (for all $t > 0$)



Theorem (Galloway–G.H. 2024)

Suppose (M, g)

1. has regular cosmological time function $\tau \colon M \to \mathbb{R}_{>0}$
2. is future timelike geodesically complete
3. contains a compact Cauchy hypersurface

Then, for every $t > 0$, the level set $\{\tau = t\}$ is a Cauchy hypersurface.

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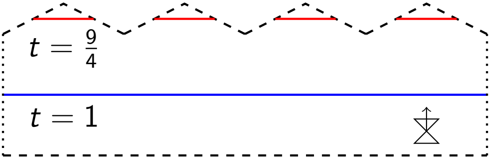
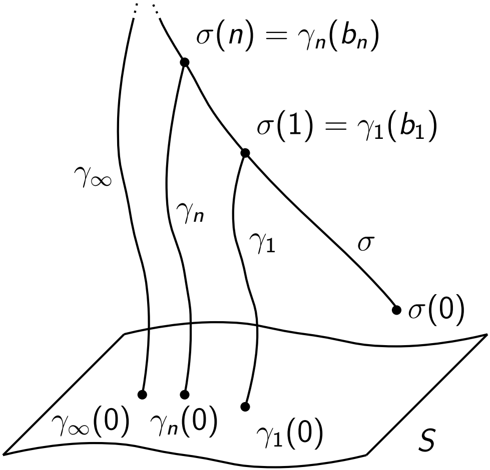
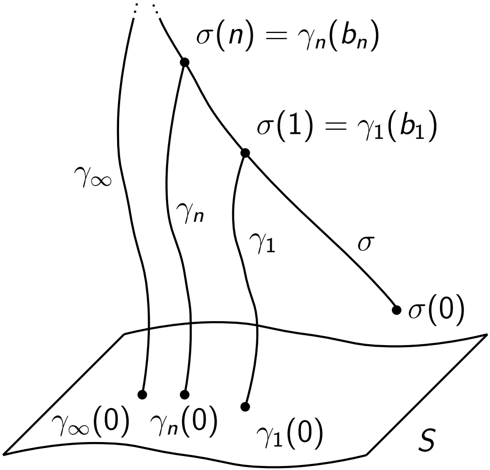
Counterexample if 2 not satisfied:





Proof. Suppose $\exists\sigma$ such that $\tau \circ \sigma \not\to +\infty$



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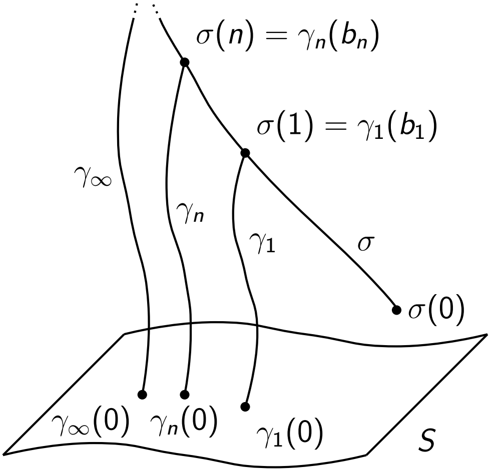

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Limit curve theorem:
$\gamma_n \to \gamma_\infty$ with
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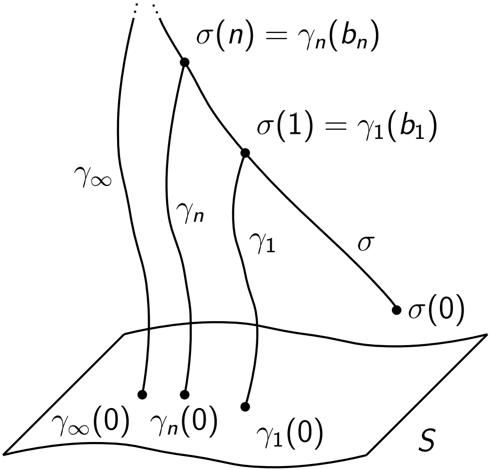

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Geodesic completeness
$\implies L(\gamma_\infty) = \infty$ ↯

Conclusion

Open questions:

- Is compact Cauchy hypersurface necessary?
- What about looking only at small times?

Applications:

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- Isometry group is compact (in progress with A. Zeghib)

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Thank you for listening!