Developments of initial data on big bang singularities for the Einstein–nonlinear scalar field equations

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Fournodavlos and Luk initial data on the singularity

Fournodavlos and Luk (2020) constructed solutions to the Einstein vacuum equations of the form

$$g = -dt \otimes dt + \sum_{i,j=1}^{3} a_{ij} t^{2p_{\max\{i,j\}}} dx^i \otimes dx^j$$

on $(0, T] \times \mathbb{T}^3$, where $p_i : \mathbb{T}^3 \to \mathbb{R}$ and $a_{ij} : (0, T] \times \mathbb{T}^3 \to \mathbb{R}$ are smooth and satisfy

$$\lim_{t \to 0^+} a_{ij}(t) = c_{ij},$$

where $c_{ij} : \mathbb{T}^3 \to \mathbb{R}$ are some prescribed smooth functions.

Related works

Constructions of quiescent big bang spacetimes:

- In symmetric settings: Isenberg, Moncrief, 1990, 2002; Kichenassamy, Rendall, 1998; Rendall, 2000; Ståhl, 2002; Choquet-Bruhat, Isenberg, Moncrief, 2004; Ames, Beyer, Isenberg and LeFloch, 2013, 2017.
- Analytic spacetimes without symmetry: Andersson, Rendall, 2001; Damour, Henneaux, Rendall, Weaver, 2002; Klinger, 2015.
- No symmetry or analyticity: Fournodavlos, Luk, 2020; Athanasiou, Fournodavlos, 2024.

Ringström (2022) introduced a geometric notion of initial data on big bang singularities.

The Einstein–nonlinear scalar field equations

Let (M, g) be a 4-dimensional spacetime, $\varphi \in C^\infty(M)$ and $V \in C^\infty(\mathbb{R})$. The equations are

$$\mathrm{Ric} - \frac{1}{2} S g + \Lambda g = T,$$
$$\Box_g \varphi = V' \circ \varphi,$$

where T is given by

$$T = d\varphi \otimes d\varphi - \left(\frac{1}{2} |d\varphi|_g^2 + V \circ \varphi \right) g.$$

Initial data on the singularity

Definition (Ringström '22)

Let $(\Sigma, \mathring{\mathcal{H}})$ be a closed 3-dimensional Riemannian manifold, $\mathring{\mathcal{K}}$ a (1,1)-tensor on Σ and $\mathring{\Phi}, \mathring{\Psi} \in C^\infty(\Sigma)$ such that:

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3. The eigenvalues of $\mathring{\mathcal{K}}$ are everywhere distinct.
4. $\gamma^1_{23} = 0$ in a neighborhood of $x \in \Sigma$ if $p_1(x) \leq 0$, where $[e_i, e_k] = \gamma^\ell_{ik} e_\ell$ and $\mathring{\mathcal{K}}(e_i) = p_i e_i$ with $p_1 < p_2 < p_3$.

Then $(\Sigma, \mathring{\mathcal{H}}, \mathring{\mathcal{K}}, \mathring{\Phi}, \mathring{\Psi})$ are initial data on the singularity.

The expansion normalized quantities

Definition

Let (M, g) be a spacetime and $\Sigma \subset M$ a spacelike hypersurface with future pointing unit normal U. Let h and k denote the induced metric and the second fundamental form of Σ. If $\mathrm{tr}_h k > 0$,

$$\mathcal{K}(X) := \frac{1}{\mathrm{tr}_h k} k(X, \cdot)^\sharp, \qquad \mathcal{H}(X, Y) := h\Big((\mathrm{tr}_h k)^{\mathcal{K}}(X), (\mathrm{tr}_h k)^{\mathcal{K}}(Y)\Big),$$

where $(\mathrm{tr}_h k)^{\mathcal{K}}(X) := \sum_{n=0}^{\infty} \frac{(\ln \mathrm{tr}_h k)^n}{n!} \mathcal{K}^n(X)$ and $X, Y \in \mathfrak{X}(\Sigma)$.

For $\varphi \in C^\infty(M)$,

$$\Psi := \frac{1}{\mathrm{tr}_h k} U\varphi, \qquad \Phi := \varphi + \Psi \ln(\mathrm{tr}_h k).$$

Development of the data

Definition

Let $(\Sigma, \mathring{\mathcal{H}}, \mathring{\mathcal{K}}, \mathring{\Phi}, \mathring{\Psi})$ be initial data on the singularity and let V be an admissible potential. Let (M, g, φ) solve Einstein's equations and suppose there is a diffeomorphism

$$F : (0, T) \times \Sigma \to U \subset M$$

such that

$$F^* g = -dt \otimes dt + h,$$

$\mathrm{tr}_h k \to \infty$ as $t \to 0$, and

$$\|\mathcal{H} - \mathring{\mathcal{H}}\|_{C^m(\Sigma)} + \|\mathcal{K} - \mathring{\mathcal{K}}\|_{C^m(\Sigma)} + \|\Phi - \mathring{\Phi}\|_{C^m(\Sigma)} + \|\Psi - \mathring{\Psi}\|_{C^m(\Sigma)} \leq C_m t^{\delta}.$$

Then (M, g, φ) is a locally Gaussian development of the data.

Existence and detailed asymptotics

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Let (M, g, φ) be a locally Gaussian development of the data. Then

$$\|t(\mathrm{tr}_h k) - 1\|_{C^m(\Sigma)} + \|\ln(\mathrm{tr}_h k) + \ln t\|_{C^m(\Sigma)} \leq C_m t^\delta.$$

Moreover,

$$F^* g = -dt \otimes dt + \sum_{i,k} b_{ik} t^{2p_{\max\{i,k\}}} \omega^i \otimes \omega^k$$

on $(0, T) \times \Sigma$, where $b_{ik} \in C^\infty((0, T) \times \Sigma)$ satisfy $b_{ik} \to \delta_{ik}$ as $t \to 0$.

Uniqueness

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Suppose there are two locally Gaussian developments of the same initial data on the singularity. Then the developments are locally isometric in a neighborhood of the singularity.

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Theorem

Given initial data on the singularity, there is a maximal locally Gaussian globally hyperbolic development of the data which is unique up to isometry.

Remark

This is analogous to the classical result by Choquet-Bruhat and Geroch!

Remarks

- Einstein's equations have a geometric singular initial value problem formulation.
- What about other types of foliations?
- From regular initial data to data on the singularity?

THANK YOU!