Hyperboloidal initial data without logarithmic singularities

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Joint work with István Rácz
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What we know

Conformal method (Andersson&Chruściel, ’93, ’94, ’96)

- polyhomogeneous expansion in $\omega = r^{-1}$

$$\lim_{\omega\to 0} \omega \log \omega = 0$$

- conditions ensuring smooth initial data
- conditions guaranteeing smooth boundary of development
- constant mean curvature

Evolutionary method (Beyer&Ritchie, 2023)

- requiring finite regularity ensures smoothness
- Bondi mass is guaranteed
- role of Bondi mass?
- $$\lim_{\omega\to 0} K_{ab}\widehat{n}^a\widehat{n}^b = const$$

Outline of talk

- extend Beyer&Ritchie by relaxing $\lim_{\omega\to 0} K_{ab}\widehat{n}^a\widehat{n}^n = const$
  - our Kerr slice actually relies on this extension
- numerical evidence
- idea of proof (rigorous proof in upcoming paper)
- discuss the role of Bondi mass
- relate results to Andersson&Chruściel

Spin-weighted variables

For $(h_{ab}; K_{ab})$ in Σ

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\begin{aligned}
{}^{(3)}R + (K^a{}_a)^2 + K_{ab}K^{ab} &= 0 \\
D_b K^b{}_a - D_a K^b{}_b &= 0
\end{aligned}
$$

ASSUME: Σ can be foliated with $\mathcal{S}_\rho$ 1-parameter family of 2-surfaces

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- $\widehat{n}_a$, $\widehat{\gamma}_{ab} = h_{ab} - \widehat{n}_a \widehat{n}_b$
- ρ^a st. $\rho^a \partial_a \rho = 1$, and $\widehat{N}$, $\widehat{N}^a$

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- introduce coordinates (ϑ, φ) and complex null dyad q^a (wrt. unit sphere metric q_{ab}) on some $\mathcal{S}_\rho$ and Lie-drag them along ρ^a

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- $(h_{ab}; K_{ab}) \quad \to$ spin-weighted functions

$$ \widehat{N} = \rho^a \widehat{n}_a \qquad \boldsymbol{\kappa} = \widehat{n}^a \widehat{n}^b K_{ab} $$
$$ \mathbf{N} = q^a \widehat{\gamma}_{ab} \rho^b \qquad \mathbf{k} = q^a \widehat{n}^b \widehat{\gamma}_a{}^c K_{bc} $$
$$ \mathbf{a} = \frac{1}{2} q^a \overline{q}^b \widehat{\gamma}_{ab} \qquad \mathbf{K} = \widehat{\gamma}^{ab} K_{ab} $$
$$ \mathbf{b} = \frac{1}{2} q^a q^b \widehat{\gamma}_{ab} \qquad \mathring{\mathbf{K}}_{qq} = q^a q^b \left(\widehat{\gamma}_a{}^c \widehat{\gamma}_b{}^d K_{cd} - \frac{1}{2} \widehat{\gamma}_{ab} \widehat{\gamma}^{cd} K_{cd} \right) $$

Parabolic-hyperbolic form of constraints

$$(h_{ab}; K_{ab}) \quad \to \quad (\widehat{N}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \mathring{\mathbf{K}}_{qq})$$

- we can solve for $(\widehat{N}, \mathbf{k}, \mathbf{K})$, free to specify $(\mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{qq})$ over Σ
  - parabolic PDE for $\widehat{N}$
  - symmetric hyperbolic system for $(\mathbf{k}, \mathbf{K})$
- provide initial data on $\mathcal{S}_0$, then integrate toward $\mathscr{I}^+$
- in numerical example
  - free data is hyperboloidal slice of Kerr $(M = 1,\ a = 1/2)$
  - ${}^{(\Delta)}\mathbf{K}\big|_{r=1} = 10 \cdot {Y_2}^0$
- no direct control over the asymptotics of solution

Asymptotically hyperboloidal data

Translated from Andersson&Chruściel, same as Prop. 1 of Beyer&Ritchie

- can add a closure to Σ
- h_{ab} and K_{ab} continuously extend
- $K = h^{ab}K_{ab}$ is bounded away from 0
- nothing on derivatives
- no constraints
- bare minimum

$$
\begin{aligned}
\widehat{N} &= \widehat{N}_1\omega + \mathscr{O}(\omega^2), \\
\mathbf{N} &= \mathscr{O}(\omega), \\
\mathbf{a} &= \omega^{-2} + \mathscr{O}(\omega^{-1}), \\
\mathbf{b} &= \mathscr{O}(\omega^{-1}), \\
\mathbf{K} &= \mathbf{K}_0 + \mathscr{O}(\omega), \\
\mathbf{K} - 2\boldsymbol{\kappa} &= \mathscr{O}(\omega), \\
\mathbf{k} &= \mathscr{O}(1), \\
\overset{\circ}{\mathbf{K}}_{qq} &= \mathscr{O}(\omega^{-1}).
\end{aligned}
$$

Asymptotic expansion

- write fundamental quantities as

$$q(\omega, \vartheta, \varphi) = \sum_{n=i}^{N} q_n(\vartheta, \varphi)\omega^n + \mathcal{O}(\omega^{n+1})$$

with q_n smooth

- insert into constraint equations

$$f_0 + f_1\omega + \ldots + f_n\omega^n + \mathcal{O}(\omega^{n+1}) = g_0 + g_1\omega + \ldots + g_n\omega^n + \mathcal{O}(\omega^{n+1})$$

  - algebraic relations from equalities of $f_0, \ldots f_n$ and $g_0, \ldots g_n$
  - system of PDEs for the residual

Hyperboloidal initial data of decent regularity

Generalization of Prop. 2 of Beyer&Ritchie

- assume free data satisfies falloff conditions ($\boldsymbol{\kappa}_0 > 0$)

$$\mathbf{N} = \mathscr{O}(\omega), \qquad \mathbf{a} = \omega^{-2} + \mathscr{O}(\omega^{-1}), \qquad \mathbf{b} = \mathscr{O}(\omega^{-1}),$$

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}_0 + \mathscr{O}(\omega), \qquad \mathring{\mathbf{K}}_{qq} = \mathscr{O}(\omega^{-1})$$

- assume solution is smooth, $\widehat{N} > 0$, and

$$\widehat{N} = \widehat{N}_0 + \widehat{N}_1\omega + \mathscr{O}(\omega^2), \qquad \mathbf{k} = \mathbf{k}_0 + \mathscr{O}(\omega), \qquad \mathbf{K} = \mathbf{K}_0 + \mathscr{O}(\omega),$$

Then

- $\widehat{N}_0 = 0,\ \widehat{N}_1 = \boldsymbol{\kappa}_0{}^{-1},\ \mathbf{K}_0 = 2\boldsymbol{\kappa}_0,\ \mathbf{k}_0 = \eth\boldsymbol{\kappa}_0/\boldsymbol{\kappa}_0$
- $(\widehat{N}, \mathbf{N}, \mathbf{a}, \mathbf{b}; \boldsymbol{\kappa}, \mathbf{k}, \mathbf{K}, \mathring{\mathbf{K}}_{qq})$ is asymptotically hyperboloidal

Smooth hyperboloidal initial data

Generalization of Prop. 3 of Beyer&Ritchie

- assume free data satisfies falloff conditions ($\boldsymbol{\kappa}_0 > 0$)

$$\mathbf{N} = \omega\widetilde{\mathbf{N}}(\omega)(= \mathcal{O}(\omega)), \qquad \mathbf{a} = \omega^{-2} + \cancel{\mathbf{a}_{-1}\omega^{-1}} + \widetilde{\mathbf{a}}(\omega),$$

$$\mathbf{b} = \cancel{\mathbf{b}_{-1}\omega^{-1}} + \widetilde{\mathbf{b}}(\omega), \qquad \boldsymbol{\kappa} = \boldsymbol{\kappa}_0 + \cancel{\boldsymbol{\kappa}_1\omega} + \omega^2\widetilde{\boldsymbol{\kappa}}(\omega),$$

$$\mathring{\mathbf{K}}_{qq} = \cancel{\mathring{\mathbf{K}}_{qq,-1}\omega^{-1}} + \mathring{\mathbf{K}}_{qq,0} + \omega\widetilde{\mathring{\mathbf{K}}}_{qq}(\omega), \qquad \overline{\eth}\left[\mathring{\mathbf{K}}_{qq,0}\cdot\boldsymbol{\kappa}_0^{-1} - \frac{1}{2}\overline{\eth}\eth\boldsymbol{\kappa}_0{}^{-2}\right] = 0$$

with $(\widetilde{\mathbf{N}}, \widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}, \widetilde{\boldsymbol{\kappa}}, \widetilde{\mathring{\mathbf{K}}}_{qq})$ being smooth

- assume solution is C^4 or C^3, $\widehat{N} > 0$, and

$$\widehat{N} = \widehat{N}_0 + \widehat{N}_1\omega + \widehat{N}_2\omega^2 + \widehat{N}_3\omega^3 + \widehat{N}_4\omega^4 + w_{\widehat{N}}(\omega)\omega^4,$$

$$\mathbf{k} = \mathbf{k}_0 + \mathbf{k}_1\omega + \mathbf{k}_2\omega^2 + \mathbf{k}_3\omega^3 + w_{\mathbf{k}}(\omega)\omega^3,$$

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_1\omega + \mathbf{K}_2\omega^2 + \mathbf{K}_3\omega^3 + \mathbf{K}_4\omega^4 + w_{\mathbf{K}}(\omega)\omega^4$$

with $(w_{\widehat{N}}, w_{\mathbf{k}}, w_{\mathbf{K}})$ being C^0 and vanishing at $\mathscr{I}^+$

Algebraic relations

$$\widehat{N}_0 = 0,\ \widehat{N}_1 = \boldsymbol{\kappa}_0^{-1},\ \widehat{N}_2 = -\tfrac{1}{2}\frac{\mathbf{K}_1}{\boldsymbol{\kappa}_0^2}\,,\ \widehat{N}_3 = -\tfrac{3}{2}\frac{(\eth\boldsymbol{\kappa}_0)(\overline{\eth}\boldsymbol{\kappa}_0)}{\boldsymbol{\kappa}_0^5} + \tfrac{1}{2}\frac{\eth\overline{\eth}\boldsymbol{\kappa}_0}{\boldsymbol{\kappa}_0^4} + \tfrac{1}{4}\frac{\mathbf{K}_1^2 - 2}{\boldsymbol{\kappa}_0^3}$$

$$+ \tfrac{1}{2}\frac{2\boldsymbol{\kappa}_2 + (\overline{\mathbf{N}}_1\eth\boldsymbol{\kappa}_0 + \mathbf{N}_1\overline{\eth}\boldsymbol{\kappa}_0)}{\boldsymbol{\kappa}_0^2} - \tfrac{1}{4}\frac{4\mathbf{a}_0 + (\eth\overline{\mathbf{N}_1} + \overline{\eth}\mathbf{N}_1)}{\boldsymbol{\kappa}_0}\,,$$

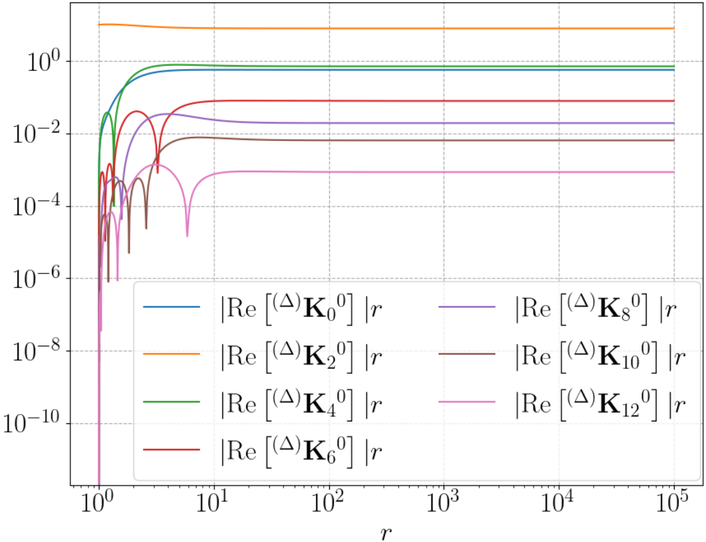
$$\mathbf{K}_0 = 2\boldsymbol{\kappa}_0,\ \mathbf{K}_1 = \mathbf{K}_1,\ \mathbf{K}_2 = \frac{\eth\boldsymbol{\kappa}_0\overline{\eth}\boldsymbol{\kappa}_0}{\boldsymbol{\kappa}_0^3} + \overline{\eth}\eth\boldsymbol{\kappa}_0^{-1} - (\overline{\mathbf{N}}_1\eth\boldsymbol{\kappa}_0 + \mathbf{N}_1\overline{\eth}\boldsymbol{\kappa}_0) - 2\boldsymbol{\kappa}_2\,,$$

$$\mathbf{k}_0 = \frac{\eth\boldsymbol{\kappa}_0}{\boldsymbol{\kappa}_0},\ \mathbf{k}_1 = \tfrac{1}{2}\frac{\boldsymbol{\kappa}_0\eth\mathbf{K}_1 - \mathbf{K}_1\eth\boldsymbol{\kappa}_0}{\boldsymbol{\kappa}_0^2},\ \mathbf{k}_2 = \mathbf{k}_2$$

- $(\widehat{N}_0, \widehat{N}_1, \mathbf{K}_0, \mathbf{k}_0)$ are completely determined by free data
- $(\mathbf{K}_1, \mathbf{k}_2, \widehat{N}_4)$ are left unconstrained $\rightarrow$ asymptotic freedom
- $(\widehat{N}_2, \widehat{N}_3, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4, \mathbf{k}_1, \mathbf{k}_3)$ are determined by free data + asymptotic freedom
- extra relation on $\overset{\circ}{\mathbf{K}}_{qq,0}$ needed to close the system

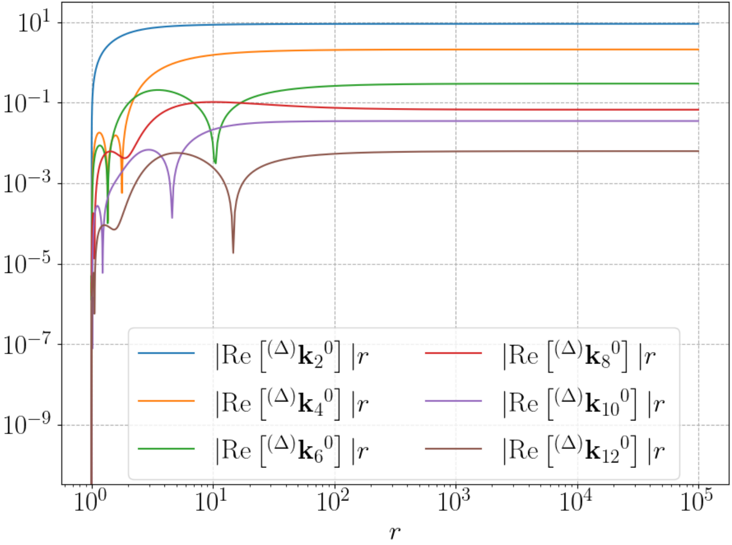
Leading order contribution of preturbations

$\mathbf{K}_0 = 2\boldsymbol{\kappa}_0$, $\mathbf{K}_1 = \mathbf{K}_1$



Leading order contribution of preturbations

$\mathbf{k}_0 = \frac{\eth \boldsymbol{\kappa}_0}{\boldsymbol{\kappa}_0}$, $\mathbf{k}_1 = \frac{1}{2} \frac{\boldsymbol{\kappa}_0 \eth \mathbf{K}_1 - \mathbf{K}_1 \eth \boldsymbol{\kappa}_0}{\boldsymbol{\kappa}_0^2}$



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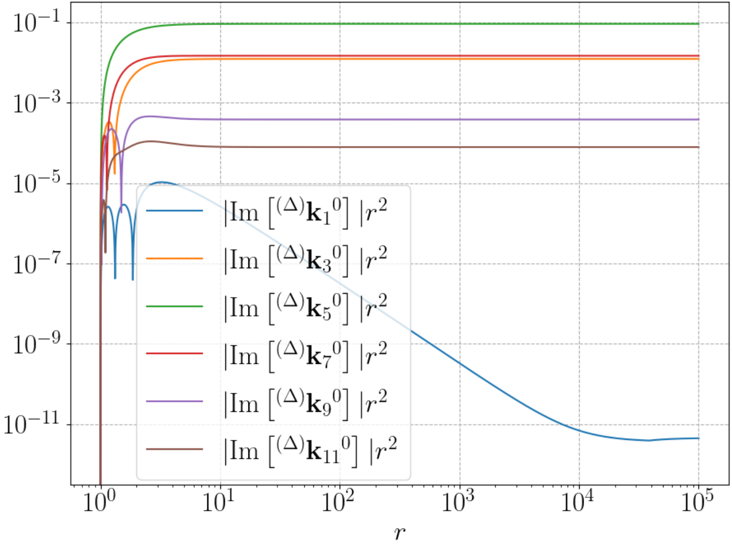
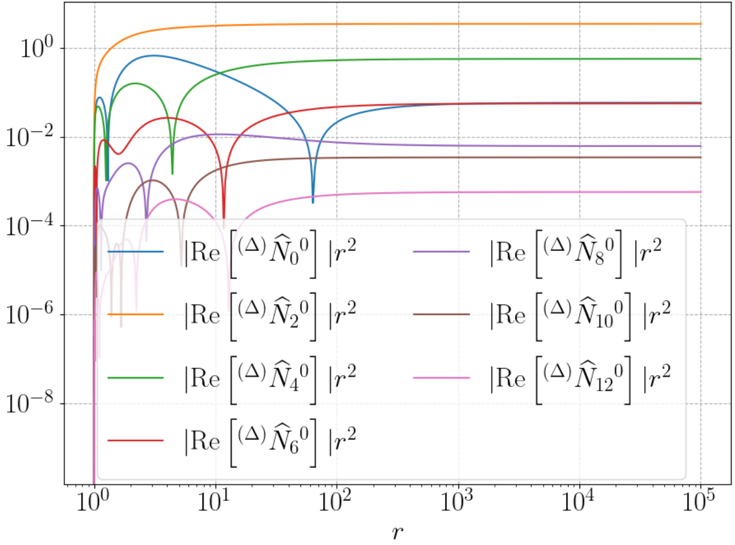
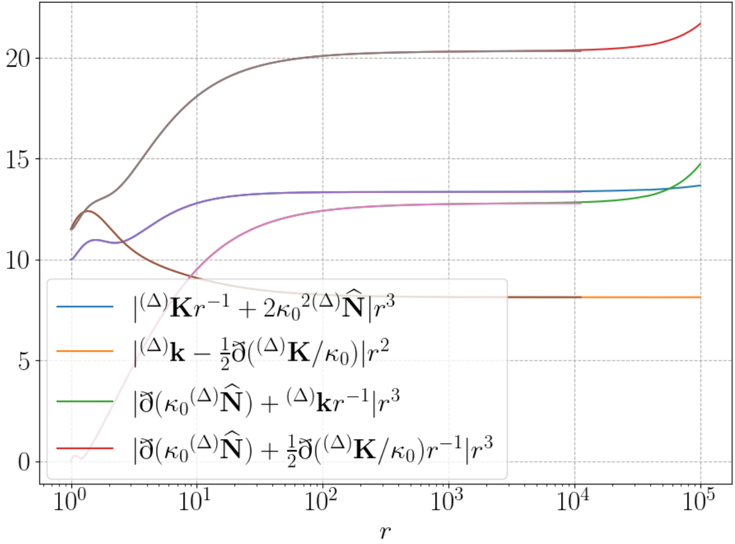


Leading order contribution of preturbations

$\widehat{N}_0 = 0,\ \widehat{N}_1 = \boldsymbol{\kappa}_0^{-1},\ \widehat{N}_2 = -\frac{1}{2}\frac{\mathbf{K}_1}{\boldsymbol{\kappa}_0^2}$



Next to leading order contribution of preturbations



Fuchsian analysis and smoothness

- PDE for the residuals

$$\partial_\omega \underline{W}(\omega,p) = \frac{1}{\omega} diag\big[-3,-1,0\big] \times \underline{W}(\omega,p)$$
$$+\, \underline{H}\big(\omega,p;\mathbf{K}_1(p),\mathbf{k}_2(p),\widehat{N}_4(p),\underline{W}(\omega,p),\eth\underline{W}(\omega,p),\bar{\eth}\underline{W}(\omega,p),\eth\bar{\eth}\underline{W}(\omega,p)\big)$$

- then the solution

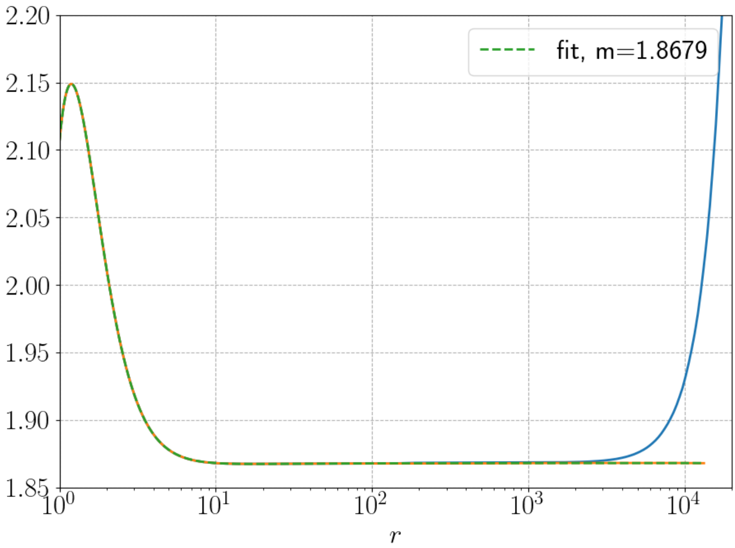
$$\underline{W}(\omega,p) = diag\big[\omega^{-3},\omega^{-1},1\big] \times \int_0^\omega diag\big[s^3,s,1\big] \times \underline{H}(s,p)\,\mathrm{d}s$$

- introducing $s = \omega \cdot \tau$

$$\frac{1}{\omega}\,\underline{W}(\omega,p) = \int_0^1 diag\big[\tau^3,\tau,1\big] \times \underline{H}(\omega\cdot\tau,p)\,\mathrm{d}\tau$$

- hence $\underline{W} = (w_{\mathbf{K}}, w_{\mathbf{k}}, w_{\widehat{N}})$ is C^∞ instead of C^0, and Bondi mass is well defined

Hawking mass



Finite limit of Hawking mass

- with $n_a^{(\pm)} = n_a \pm \widehat{n}_a$

$$m_H = \sqrt{\frac{\mathcal{A}}{16\pi}} \left(1 + \frac{1}{16\pi} \int_{\mathscr{S}_r} \left[\mathbf{K}^2 - \overset{\star}{K}{}^2 \widehat{N}^{-2} \right] \sqrt{\mathbf{d}}\, \epsilon_q \right)$$

- since $\mathcal{A} \sim \mathcal{O}(r^2)$ we need $\left[\mathbf{K}^2 - \overset{\star}{K}{}^2 \widehat{N}^{-2} \right] \sqrt{\mathbf{d}} \sim -4 + \mathcal{O}(r^{-1})$
- actually $\left[\mathbf{K}^2 - \overset{\star}{K}{}^2 \widehat{N}^{-2} \right] \sqrt{\mathbf{d}} = f_{-2}\omega^{-2} + f_{-1}\omega^{-1} + f_0 + f_1\omega + \mathcal{O}(\omega^2)$
- log terms in $\mathbf{K}$ and $\widehat{N}$ are ruled out
  - conjecture: angular momentum rules out log terms in $\mathbf{k}$
- constraints give $\boldsymbol{\kappa}_1 = 0$ and make $f_{-2} = 0$
- vanishing of f_{-1} gives $\mathbf{a}_{-1} = 0$
- $f_0 = -4$ gives $\mathbf{b}_{-1} = 0$ and $\overset{\circ}{\mathbf{K}}_{qq,-1} = 0$

Smooth boundary of Cauchy development

- conditions of Cauchy development having a smooth boundary (Andersson&Chruściel)
  - $\left.\widetilde{K}^{log}_{rr}\right|_{\mathscr{I}+} = 0$

$$\widetilde{K}^{log}_{rr} = \mathbf{K}^{log}_{1} + \omega\left[\mathbf{K}^{log}_{2} + 2\,\boldsymbol{\kappa}_0\,\widehat{N}_1\,\widehat{N}^{log}_{1}\right] + \mathscr{O}(\omega^2)$$

  - $\left.\widetilde{K}^{log}_{rA}\right|_{\mathscr{I}+} = 0$

$$\widetilde{K}^{log}_{ra} = \tfrac{1}{2}\left[\mathbf{k}_0\,\overline{q}_a + \overline{\mathbf{k}}_0\,q_a\right]\widehat{N}^{[log]}_{1} + \mathcal{O}(\omega)$$

  - $\left.\omega[\mathring{\mathbf{K}}_{qq} - \mathring{\widehat{K}}_{qq}]\right|_{\mathscr{I}+} = 0$

$$\mathring{\mathbf{K}}_{qq} - \mathring{\widehat{K}}_{qq} = \left[\mathring{\mathbf{K}}_{qq(-1)} + \boldsymbol{\kappa}_0\mathbf{b}_{(-1)}\right]\omega^{-1} + \mathcal{O}(1)$$

- well defined Bondi mass and angular momentum ensures this
- do we miss out on anything by using the strickter condition?

Summary

- generalized Beyer&Ritchie to accomodate Kerr
- requiring finite Bondi mass explains almost all conditions of Beyer&Ritchie
- requiring finite Bondi mass ensures that Cauchy development has a smooth boundary
- code available under MIT license at `https://gitlab.wigner.hu/csukas.karoly/constraintsolver`