Importance of non-linearities in a black-hole ringdown

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based on arXiv:2405.10270

in collaboration with
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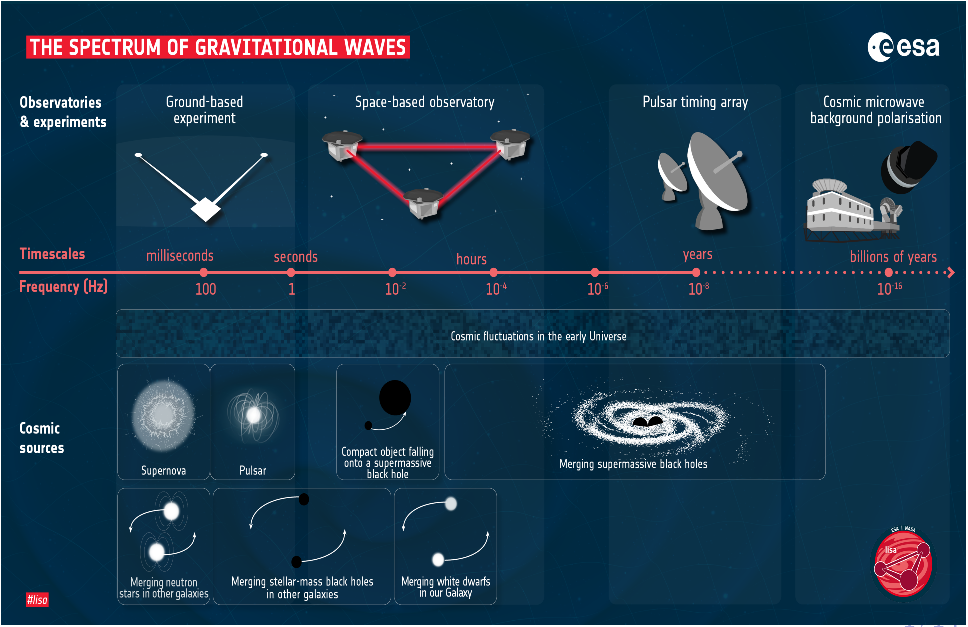
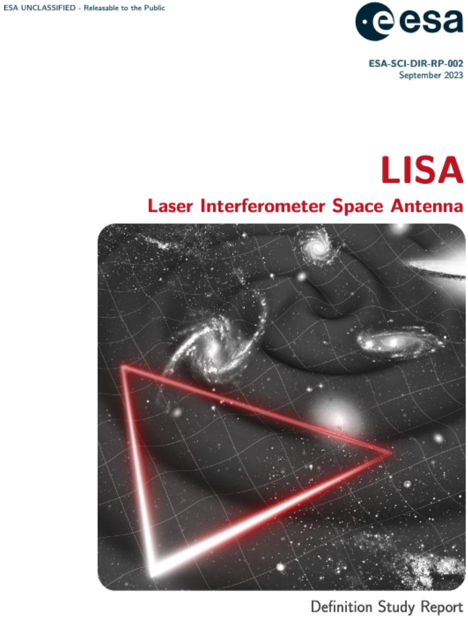
The gravitational-wave spectrum



LISA - Definition Study Report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LISA

Laser Interferometer Space Antenna

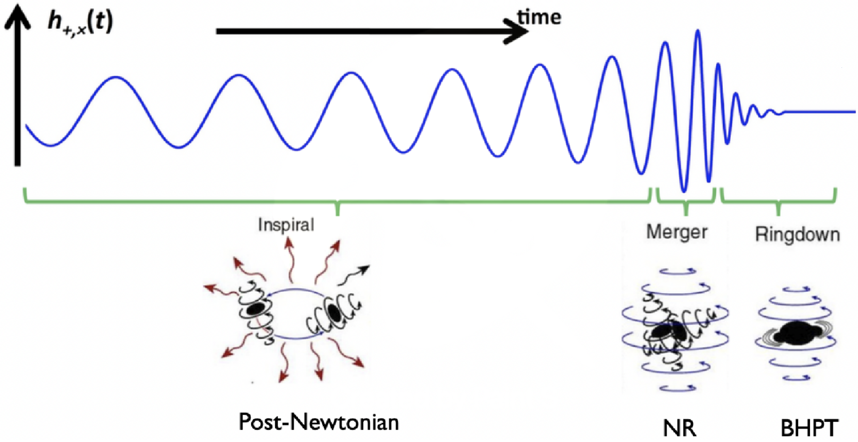
Definition Study Report

LISA - Mission summary

Science Objectives

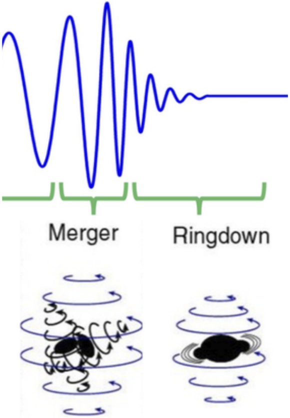
- Study the formation and evolution of **compact binary stars** and the structure of the Milky Way Galaxy
- Trace the origins, growth and merger histories of **massive Black Holes** across cosmic epochs
- Probe the properties and immediate environments of Black Holes in the local Universe using **extreme mass-ratio inspirals** and **intermediate mass-ratio inspirals**
- Understand the astrophysics of **stellar-mass Black Holes**
- Explore the **fundamental nature of gravity** and Black Holes
- Probe the rate of **expansion of the Universe** with standard sirens
- Understand **stochastic gravitational wave backgrounds** and their implications for the early Universe and TeV-scale particle physics
- Search for gravitational wave bursts and **unforeseen sources**

Gravitational waveform signal from inspiral binaries



Credit: Marc Favata/SXS/Kip Thorne

Ringdown: Quasi-normal modes



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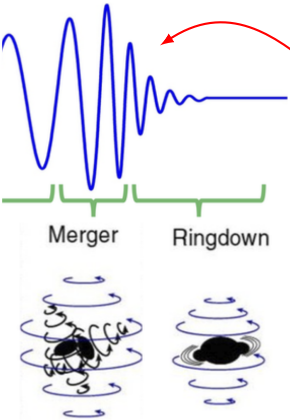

Ringdown signal: superposition of complex frequencies

$$h(t) = \sum_{k=0}^{\infty} A_k e^{-i\omega_k t}$$

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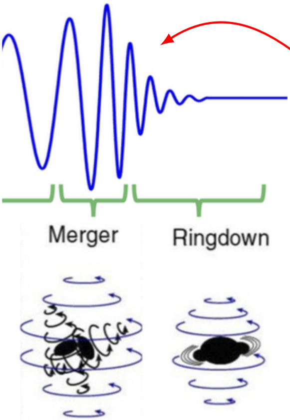


Ringdown signal: superposition of complex frequencies

$$h(t) = \sum_{k=0}^{\infty} A_k e^{-i\omega_k t}$$

but from perturbation theory:
$\omega_k = \omega_{\ell mn}^{QNM}$ (quasi-normal modes)

$$\implies h(t) = \sum_{\ell mn}^{\infty} A_{\ell mn} e^{-i\omega_{\ell mn} t}$$

Black-hole spectroscopy

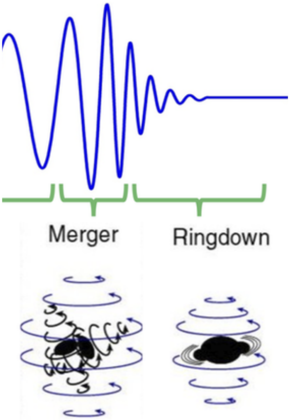
Main point: Each $\omega_{\ell mn}$ depend only on the Kerr parameters!

$$\omega_{\ell mn} = \omega_{\ell mn}(M_f, a_f)$$

The detection of a single QNM mode (for example the $(\ell, m, n) = (2, 2, 0)$) gives an estimate of (M_f, a_f)!

$\implies$ **Black-hole spectroscopy**: the detection of multiple QNMs can be used to test if the remnant is a Kerr black-hole and the no-hair theorem.

QNM data analysis — first-order



When does ringdown start?

- some $\propto M$ after the peak?
- Which peak? h_+? $h_\times$? Ψ_4?

Other issues

- QNMs form an incomplete set!
- QNM frequencies are unstable: spectral instability
- Growing evidence of detectable non-linear effects

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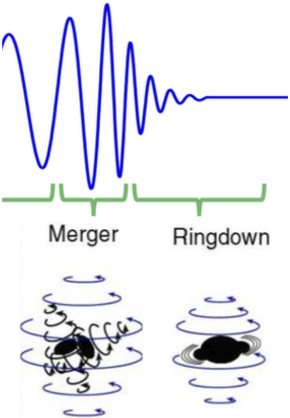

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Quasi-normal modes at second-order

Use Teukolsky equation.

$$\mathcal{O}(\Psi_4^{(1)}) = 0$$

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Can we make this more precise?

How linear mode excitation contribute at second-order

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Issue: different papers find different values of this ratio $\mathcal{R}$!?

Regular vs mirror modes

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In reality, the source term is actually made up of **two** different kinds of products:

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regular mode — complex conjugation — mirror mode

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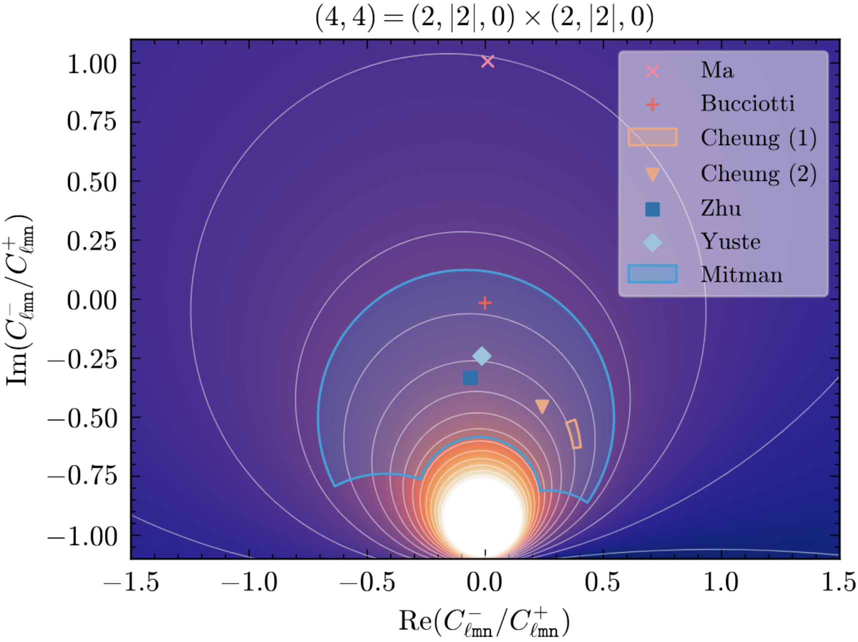
related to odd/even mode amplitude, $C^{-}_{\ell mn}/C^{+}_{\ell mn}$

QQNM dependence on linear mode parity



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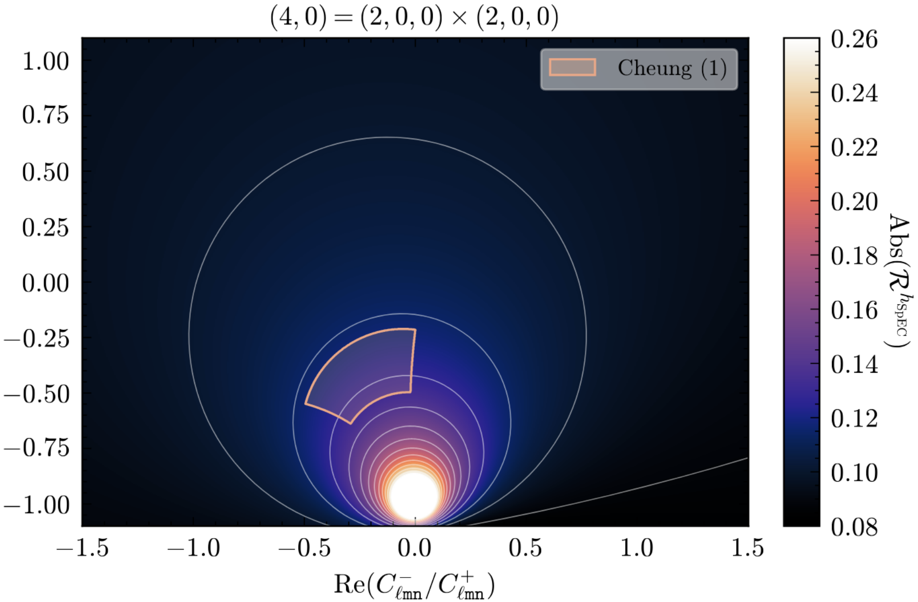

Conclusion

- With future space-based GW detectors, such as LISA, we can explore yet uncharted territories.
- The black-hole spectroscopy program aims at establishing if the merger remnant is Kerr and test the no-hair theorem.
- Conclusive evidence shows that precise measurement of the waveform ringdown must account for non-linear effect.
- We showed the second-order contribution depends on the ratio of even to odd linear parity modes.
- This dependence has been historically overlooked and can be used to obtain better error estimates of the ratio of second-order to parent linear-order modes.

Future directions:

- Generalise the formalism to Kerr.
- Investigate second-order effects at the horizon
- Investigate branch cut contribution.