A new proof of the extended Minkowski inequality via a divergence inequality

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Historical Background

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Theorem

Let K_1 be a convex body[a] and K_2 a ball of radius 1. Denote by V_0 the volume, $3V_1$ the surface area, and $3V_2$ the integral of mean curvature of K_1 and denote by $V_3 = 4\pi/3$ the volume of the unit ball in three dimensions, then

$$V_1^2 \geq V_0 V_2 \quad \text{and} \quad V_2^2 \geq V_1 V_3,$$

with equality if and only if K_1 is a ball[b].

[a] Compact convex set with non-empty interior.
[b] This is the rigidity case.

Understanding the Minkowski Inequality

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This Minkowski inequality asserts

> *Among all convex bodies with the same surface area, balls alone minimize the integral of mean curvature.*

Generalizing to higher dimensions

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- This early version of the Minkowski inequality for $n = 3$ can be directly generalized to higher dimensions $n \geq 3$.
- In modern notation the Minkowski inequality reads:

Theorem (Minkowski Inequality)

If $\Omega \Subset \mathbb{R}^n$ with $n \geq 3$ is a convex domain with smooth, boundary and H the mean curvature of $\partial\Omega$ computed with respect to the outward unit normal, then

$$\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{1/(n-1)} \leq \fint_{\partial\Omega} \frac{H}{n-1} d\sigma$$

with equality if and only if Ω is a ball.

Remark: $\fint_E f \, d\mu = \frac{1}{\mu(E)} \int_E f \, d\mu =$ "average of f over set E".

On the generalization to non-convex domains

Natural Question: Does the Minkowski Inequality hold true for larger classes of domains than just for the convex one?

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- Using the the method of smooth *Inverse Mean Curvature Flow (IMCF)*[1] Guan-Li '09 based on results from Gerhardt 1990, and Urbars 1990 extended it to the family of *starshaped domains with strictly mean-convex boundary*

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Answer: Yes!

- Using the the method of smooth *Inverse Mean Curvature Flow (IMCF)*[1] Guan-Li '09 based on results from Gerhardt 1990, and Urbars 1990 extended it to the family of *starshaped domains with strictly mean-convex boundary*
- Using the method of *Optimal Transport* Qiu '15 based on Chang-Wang '13 extended it to *bounded open sets with smooth boundary*.

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The L^p-Minkowski Inequality

One generalization of the Minkowski inequality is:

Theorem (L^p-Minkowski Inequality, Agostiniani–Fogagnolo–Mazzieri '22)

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth (connected) boundary. Then, for every $1 < p < n$, the following inequality holds

$$C_p(\Omega)^{\frac{n-p-1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left| \frac{H}{n-1} \right|^p d\sigma.$$

Here, $C_p(\Omega)$ is the normalized p-capacity of Ω and H is the mean curvature of $\partial\Omega$ computed with respect to the outward unit normal. Moreover, equality holds iff Ω is a ball.

$$\mathrm{C}_p(\Omega) = \inf \left\{ \left(\frac{p-1}{n-p} \right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} |Dv|^p d\mu \;\middle|\; v \in C_0^\infty(\mathbb{R}^n), v \geq 1 \text{ on } \Omega \right\}$$

Finding the extended Minkowski inequality

Letting $p \to 1^+$ in the L^p-Minkowski inequality and using that

$$\lim_{p \to 1^+} C_p(\Omega)^{\frac{n-p-1}{n-p}} = \left(\frac{|\partial \Omega^*|}{|\mathbb{S}^{n-1}|} \right)^{\frac{n-2}{n-1}}$$

holds, we find

Extended Minkowski Inequality

Theorem (Extended Minkowski inequality, Agostiniani–Fogagnolo–Mazzieri '22)

Let $n \geq 3$, if $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth (connected) boundary, then

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\left|\frac{H}{n-1}\right| d\sigma,$$

where Ω^* is the strictly outward minimising hull of Ω.

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On the Proof of Agostiniani–Fogagnolo–Mazzieri

- Moser '05 establishes a connection between IMCF and the problem of p-harmonic functions.
- Agostiniani, Fogagnolo and Mazzieri replace the *weak IMCF* with a novel analysis of *p-capacitary potentials* of Ω to prove the extended Minkowski inequality. These p-capacitary potentials are the weak solutions to the non-linear problem

$$\begin{cases} \Delta_p u := \operatorname{div}\left(|Du|^{p-2} Du\right) = 0 & \text{in} \quad \mathbb{R}^n \setminus \overline{\Omega} \\ u = 1 & \text{on} \quad \partial\Omega \\ u(x) \to 0 & \text{as} \quad |x| \to \infty, \end{cases} \tag{1}$$

  with $1 < p < n$.
- Their proof relies on discovering *effective monotonicity formulas* for newly constructed functionals, holding along the level set flow of the p-capacitary potentials u associated with Ω.

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5. Ongoing work of Cederbaum and León Quirós to use this approach to show the Wilmore inequality for Riemannian manifolds with non-negative Ricci curvature.
6. All these proofs use solutions to the linear Laplace equation. This is the first time solutions to the non-linear p-Laplace equation are used.

L^p-Minkowski inequality via divergence inequality I

Theorem (Divergence inequality (part I))

Let $n \geq 3$, $1 < p < n$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with smooth, connected boundary* $\partial\Omega$. *Let* u *be a* p*-capacitary potential associated with* Ω. *Set*

$$a := \begin{cases} \frac{(p-1)^3}{4(n-1)} & , \text{ if } (p-1)^2 \leq n-1 \\ \frac{p-1}{4} & , \text{ if } (p-1)^2 > n-1 \end{cases}, \quad b := \frac{(p-1)(p-2)}{4}.$$

Then the divergence inequality

$$\begin{aligned} &\operatorname{div}\left(F(u)(D|Du|^{p-1} + (p-2)D^{\perp}|Du|^{p-1}) + G(u)|Du|^{p-1}Du\right) \\ &\geq aF(u)|Du|^{p-5}\left|D|Du|^2 - \frac{2(n-1)|Du|^2}{(n-p)u}Du\right|^2 \\ &\quad + bF(u)|Du|^{p-5}\left|D^{\top}|Du|^2\right|^2 \end{aligned}$$

L^p-Minkowski inequality via divergence inequality II

Theorem (Divergence Inequality (part II))

holds on $\mathbb{R}^n \backslash \overline{\Omega}$ *for smooth functions* $F, G : (0,1] \to \mathbb{R}$ *given by*

$$F(u) = (cu + d)u^{-\frac{n-1}{n-p}+1},$$
$$G(u) = (p-1)^2\left[-\frac{n-1}{(n-p)u}F(u) + du^{-\frac{n-1}{n-p}}\right],$$

for any $c, d \in \mathbb{R}$ *satisfying* $c + d \geq 0$ *and* $d \geq 0$*. Here,* div *denotes the (Euclidean) divergence. Moreover, if* $\mathbf{p > 2}$ *we have*

$$|Du|^6 \operatorname{div}\left(F(u)(D|Du|^{p-1} + (p-2)D^{\perp}|Du|^{p-1}) + G(u)|Du|^{p-1}Du\right) \geq 0,$$

where equality holds if and only if Ω *is a round ball (unless* $c = d = 0$*).*

Outline of the proof of the divergence inequality

1. Calculate divergence of vector field ansatz

$$W := F(u)(D|Du|^{p-1} + (p-2)D^{\perp}|Du|^{p-1}) + G(u)|Du|^{p-1}Du.$$

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3. Find set of coupled ODE's for F and G to make the lower bound non-negative

$$\begin{aligned}(p-1)^2F'(u) + G(u) &= -a\frac{8(n-1)}{(n-p)u}F(u)\\ G'(u) &= a\frac{4(n-1)^2}{(n-p)^2u^2}F(u).\end{aligned}$$

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Integral identity

Proposition (Integral identity)

Let $\Omega \subset \mathbb{R}^n$ *be a bounded domain with smooth, connected boundary* $\partial\Omega$ *and p-capacitary potential* u*. Let* $c, d \in \mathbb{R}$ *such that* $c + d \geq 0$ *and* $d \geq 0$*. Consider the vector field* W*, with* F *and* G *given as before. Let* $0 < u_0 < u_1 \leq 1$*. Then*

$$\int_{\{u_0 < u < u_1\}} \operatorname{div} W d\mu = \int_{\{u = u_1\}} ((p-1)F(u)|Du|^{p-1}H - G(u)|Du|^p) d\sigma$$
$$- \int_{\{u = u_0\}} ((p-1)F(u)|Du|^{p-1}H - G(u)|Du|^p) d\sigma$$

holds.

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$$\int_{\{u_0<u<u_1\}} \operatorname{div} W d\mu = \int_{\{u=u_1\}} ((p-1)F(u)|Du|^{p-1}H - G(u)|Du|^p)d\sigma - \int_{\{u=u_0\}} ((p-1)F(u)|Du|^{p-1}H - G(u)|Du|^p)d\sigma$$

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To prove this simply

1. calculate $\langle W, \nu \rangle$ with ν the unit normal to the level set,

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3. Calculate the asymptotic behavior of $F(u)$, $G(u)$, H, $|Du|$ and $d\sigma$.

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3. Calculate the asymptotic behavior of $F(u)$, $G(u)$, H, $|Du|$ and $d\sigma$.
4. Use the asymptotics to find...

Parametric Geometric Inequality

Theorem (Parametric geometric inequality)

Let $n \geq 3$, $\mathbf{2} < \mathbf{p} < \mathbf{n}$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth connected boundary $\partial\Omega$. Let u be a p-capacitary potential associated with Ω and consider parameters $c, d \in \mathbb{R}$ satisfying $c + d \geq 0$ and $d \geq 0$. One then has

$$d(p-1)\left(\frac{n-p}{p-1}\right)^p C_p(\Omega)^{\frac{n-p-1}{n-p}} |\mathbb{S}^{n-1}| \leq (c+d)\int_{\partial\Omega} |Du|^{p-1} H d\sigma + (p-1)\left[d - \frac{n-1}{n-p}(c+d)\right]\int_{\partial\Omega} |Du|^p d\sigma.$$

Equality holds iff Ω is a round ball (unless $c = d = 0$).

Concluding the Minkowski Inequality

Choosing $c = 1$ and $d = 0$ in the parametric geometric inequality yields

$$\frac{p-1}{n-p}\int_{\partial\Omega}|Du|^p d\sigma \leq \int_{\partial\Omega}|Du|^{p-1}\left|\frac{H}{n-1}\right| d\sigma.$$

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By the Hölder inequality, one gets

$$\int_{\partial\Omega}|Du|^p d\sigma \leq \left(\frac{n-p}{p-1}\right)^p \int_{\partial\Omega}\left|\frac{H}{n-1}\right|^p d\sigma.$$

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Choosing $c = -1$ and $d = 1$ on the other hand one obtains

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Combining these two yields the L^p-Minkowski inequality.

The End

Thank you for your attention!