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Geodesic Motion in General Relativity

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Introduction and Motivation

In general relativity, the equivalence principle motivates the hypothesis that freely-falling test particles follow geodesics. At first glance, this geodesic hypothesis seems to be independent of the central equation of general relativity, the Einstein field equation. However, here we will explore two proofs which show that geodesic motion is indeed a consequence of Einstein's equation. The mathematical statement and proof of this hypothesis gives insight on the concept of point particles in general relativity.

Ehlers-Geroch

Here we consider an extension given by Bezares et al. of the original theorem. This is a proof by contradiction which shows that if we have matter supported on a world tube of a timelike curve, and we suppose this world tube can be shrunk down to the curve, then the curve must be a geodesic in the background metric.

Assumptions

1. There exists a sequence g_n of Lorentzian metrics in $\mathcal{M}(U)$ (the space of smooth metrics on $U \subset M$) such that g_n converges to g in the topology induced by the metric d_{C^1}.
2. Each metric g_n has an Einstein tensor ${}^{(n)}G_{ab}$ satisfying ${}^{(n)}\nabla_c\, {}^{(n)}G_{ab} = 0$. We assume that ${}^{(n)}G_{ab} \neq 0$ near γ and that its support, $\mathrm{spt}({}^{(n)}G_{ab})$, is contained in U_n (a tubular neighborhood of γ of radius $1/n$).
3. There exists a foliation of U by spacelike hypersurfaces.
4. For every n the tensor ${}^{(n)}G_{ab}$ satisfies an "averaged dominant energy condition" meaning along each g_n-timelike curve $\tilde{\gamma}$ near γ there exists a constant $K_n(\tilde{\gamma})$ such that $\int_{\tilde{\gamma}} \max_{ab} |{}^{(n)}G_{ab}|\ {}^{(n)}\mathrm{d}\tilde{\gamma} \leq K_n(\tilde{\gamma}) \int_{\tilde{\gamma}} {}^{(n)}G_{ab} t^a t^b\ {}^{(n)}\mathrm{d}\tilde{\gamma}$

Lemma 1 (Properties of x, t, β).

Let γ be a timelike curve with unit tangent vector field u on a spacetime (M, g), and define $A = \nabla_u u$. Assume that $A(p_0) \neq 0$ for some $p_0 \in \gamma$. Then, there exists vector fields t, x, β near p_0 such that:
At the point p_0 we define $t^a = u^a, x^a = \frac{A^a}{|A|}, \beta^a = 0$. They are transported along the curve according to $\nabla_u t = 0, \nabla_u x = 0, \nabla_u \beta^a = u_b(x^b t^a - x^a t^b)$. They are Killing along γ and $\beta = \phi t + \psi x$. Moreover, there exist points p_- and p_+ on either side of p_0 such that $\phi(p_\pm) > 0$ and $\psi(p_+) = -\psi(p_-)$. See Figure 1.

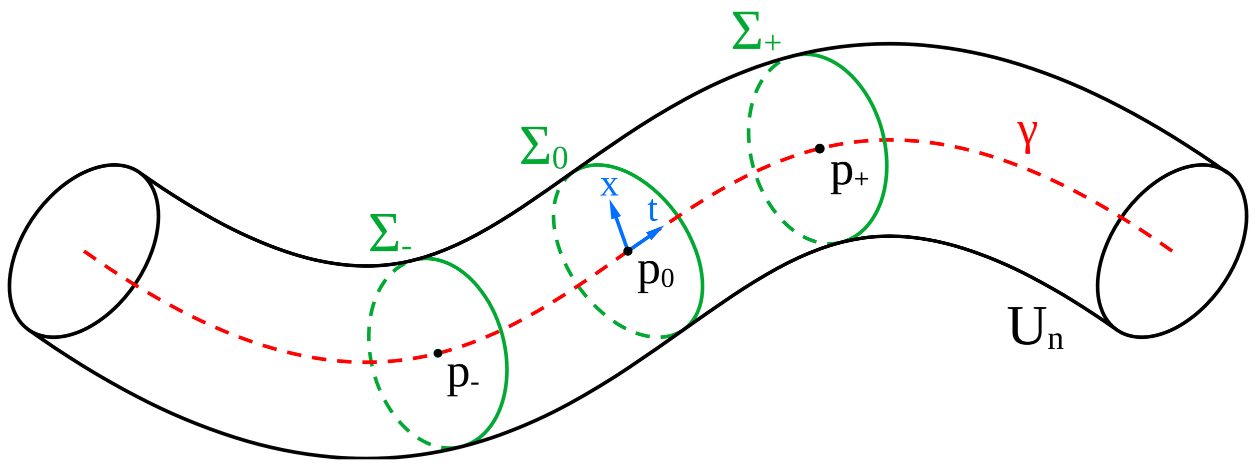

Figure 1: World tube around γ.

Theorem 2 (Ehlers-Geroch).

Suppose the above assumptions hold. Furthermore assume that $K_n(\tilde{\gamma})$ is bounded independently of n and that the following "stabilized ratio condition holds"
$\limsup_{n\to\infty} \frac{\int_\Sigma {}^{(n)}G_{ab} v^a \eta^b\ {}^{(n)}\mathrm{d}\Sigma}{\int_{\Sigma'} {}^{(n)}G_{ab} w^a \eta^b\ {}^{(n)}\mathrm{d}\Sigma'} < \infty$ *for all g_n-spacelike hypersurfaces Σ and Σ', any vector v^a, and any future-directed g_n-timelke vector w^a, where η^b is the g_n-unit-normal vector field. Then the curve γ is a geodesic.*

Outline of Proof

Aiming for a contradiction, suppose there is a point p_0 as in the lemma. An important quantity in the proof is the total energy-momentum flux along a vector field ξ through a spacelike slice Σ of U.

$$P(\xi, \Sigma) := \int_\Sigma G_{ab} \xi^b \eta^a \,\mathrm{d}\Sigma$$

Also define $m_n := P_{g_n}(t, \Sigma_n)$ where Σ_0 is the leaf of the foliation through p_0. This function has three useful properties.

1. We have $\lim_{n\to\infty} \frac{1}{m_n} |P_n(t, \Sigma) - m_n| = 0$.
2. If ξ is a Killing vector field for g along γ, then $\lim_{n\to\infty} \frac{|P_n(\xi, \Sigma_1) - P_n(\xi, \Sigma_2)|}{m_n} = 0$.
3. If a vector field ξ vanishes on $\Sigma \cap \gamma = p$ then $\lim_{n\to\infty} \frac{|P_n(\xi, \Sigma)|}{m_n} = 0$.

Now we define a particular sum $\mathcal{K}_n$ of the total energy fluxes along the vector fields x, t, β through the surfaces $\Sigma_+, \Sigma_0, \Sigma_-$ and we use these properties to estimate the limit $\lim_{n\to\infty} \frac{\mathcal{K}_n}{m_n}$ in two different ways. One way gives $\lim_{n\to\infty} \frac{\mathcal{K}_n}{m_n} = 0$ whereas the other gives $\lim_{n\to\infty} \left|\frac{\mathcal{K}_n}{m_n} + \phi_+ + \phi_-\right| = 0$, which gives a contradiction.

Gralla-Wald

The key idea of this proof is to consider a 1-parameter family of metrics $g_{ab}(\lambda)$, where the parameter λ is chosen to capture the idea of a body shrinking to a curve in a self-similar way as $\lambda \to 0$. We then consider the Einstein equation to first order in λ.

Assumptions

1. Existence of the "ordinary limit": $g_{ab}(\lambda)$ is such that there exists coordinates x^α such that $g_{\mu\nu}(\lambda, x^\alpha)$ is jointly smooth in (λ, x^α), at least for $r > \bar{R}\lambda$. For all λ and $r > \bar{R}\lambda$, $g_{ab}(\lambda)$ is a solution to Einstein's equation. Furthermore, $g_{\mu\nu}(\lambda = 0, x^\alpha)$ is smooth in x^α, including at $r = 0$, and, for $\lambda = 0$, the curve γ defined by $r = 0$ is timelike.
2. Existence of the "scaled limit": for each t_0, define $\bar{t} \equiv (t - t_0)/\lambda, \bar{x}^i \equiv x^i/\lambda$. Then the metric $\bar{g}_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{x}^\alpha) \equiv \lambda^{-2} g_{\bar{\mu}\bar{\nu}}(\lambda; t_0; \bar{x}^\alpha)$ is jointly smooth in $(\lambda, t_0; \bar{x}^\alpha)$ for $\bar{r} \equiv r/\lambda > \bar{R}$.
3. Uniformity condition: each component of $g_{ab}(\lambda)$ in coordinates x^μ is a jointly smooth function of the variables $(\alpha \equiv r, \beta \equiv \lambda/r)$ at $(0, 0)$ and fixed t, θ, ϕ.

Lemma 3 (Far-zone Expansion).

The uniformity assumption allows us to expand $g_{ab}(\lambda)$ in a Taylor series in the variables (α, β) around $(0, 0)$. The lowest order terms are explicitly

$$\begin{aligned} g_{\alpha\beta}(\lambda) =& (a_{\alpha\beta})_{00}(t) + (a_{\alpha\beta})_{10}(t, \theta, \phi) r + O(r^2) \\ &+ \lambda \left[(a_{\alpha\beta})_{01}(t, \theta, \phi) \frac{1}{r} + (a_{\alpha\beta})_{11}(t, \theta, \phi) + O(r) \right] + O(\lambda^2) \end{aligned}$$

We choose coordinates where $g_{\alpha\beta}(\lambda = 0, x^i = 0) = (a_{\alpha\beta})_{00}(t) = \eta_{\alpha\beta}$. Then our expansion has the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + O(r) + \lambda h_{\alpha\beta} + O(\lambda^2)$$

where $h_{\alpha\beta} = \frac{c_{\alpha\beta}(t, \theta, \phi)}{r} + O(1)$.

Theorem 4.

Let $g_{ab}(\lambda)$ be any one-parameter family of metrics satisfying the assumptions. Then to first order in λ, the far-zone metric perturbation h_{ab} corresponds to a solution to the linearized Einstein equation with a point particle source.
$T_{ab} = M u_a u_b \frac{\delta^{(3)}(x^i)}{\sqrt{-g}} \frac{\mathrm{d}\tau}{\mathrm{d}t}$ where M is a constant and u^a is the 4 velocity of γ, which must be a geodesic if $M \neq 0$.

Outline of Proof

From our first assumption h_{ab} is a solution to the linearized vacuum Einstein equation, with linearization taken around $g_{ab}(\lambda = 0)$, for $r > 0$. In fact, each component $h_{\alpha\beta}$ is a locally L^1 function, including at $r = 0$. Thus h_{ab} is well defined as a distribution. This lets us define $T_{ab} \equiv G^{(1)}_{ab}[h_{cd}]/8\pi$ as a distribution as well. Its action on a test tensor field is given by

$$8\pi T(f) = \int_M G^{(1)}_{ab}[f_{cd}] h^{ab} \sqrt{-g}\, \mathrm{d}^4 x$$

Evaluating this integral over a region $r > \epsilon > 0$ and taking the limit as $\epsilon \to 0$ yields

$$T(f) = \int \mathrm{d}t N_{ab}(t) f^{ab}(t, r = 0)$$

Where $N_{ab}(t)$ is a smooth, symmetric tensor field on γ whose components are given in terms of angular averages of $c_{\alpha\beta}$ and its first angular derivatives. Now notice that $\nabla^a T_{ab} = 0$ implies $T(f)$ vanishes if $f_{ab} = \nabla_{(a} f_{b)}$, Furthermore, by careful choice of test functions we find $N_{ab} = M(t) u_a u_b$, putting these observations together we conclude

$$0 = \int \mathrm{d}t M(t) u_b (u_a \nabla^a f^b)$$

Integrating by parts and using that f^b is arbitrary gives $u^a \nabla_a (M(t) u_b) = 0$ and $\frac{\mathrm{d}M}{\mathrm{d}t} = 0$, thus if $M \neq 0$ we have $u^a \nabla_a u^b = 0$, i.e. γ is a geodesic of $g_{ab}(\lambda = 0)$.

Outlook and Perspectives

We have seen two proofs of geodesic motion. The Ehlers-Geroch proof makes explicit conditions on the energy content and is expressed in a coordinate-free way, but is by contradiction not construction. On the other hand, the Gralla-Wald proof provides a perturbative construction of the trajectory of a body which allows us to study the dynamics of extended bodies by considering higher order terms. Peter Hintz has recently used gluing to construct a spacetime that satisfies the Gralla-Wald assumptions.

[1] S. E. Gralla and R. M. Wald, "A rigorous derivation of gravitational self-force," Class. Quantum Grav., vol. 25, no. 20, p. 205009, Oct. 2008, doi: 10.1088/0264-9381/25/20/205009. [2] J. Ehlers and R. Geroch, "Equation of motion of small bodies in relativity," Annals of Physics, vol. 309, no. 1, pp. 232–236, Jan. 2004, doi: 10.1016/j.aop.2003.08.020. [3] M. Bezares, G. Palomera, D. J. Pons, and E. G. Reyes, "The Ehlers–Geroch theorem on geodesic motion in general relativity," Int. J. Geom. Methods Mod. Phys., vol. 12, no. 03, p. 1550034, Mar. 2015, doi: 10.1142/S0219887815500346.