Local Well-posedenss of the Bartnik Static Extension Problem near Schwarzschild spheres

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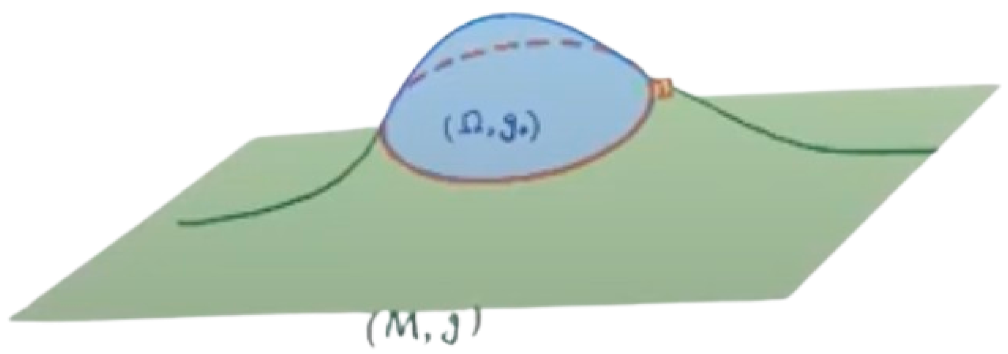
Bartnik's Quasi-local mass (1989)

Let (Ω, g_0) be a compact 3-manifold with boundary Σ. Suppose $R_{g_0} \geq 0$. The *Bartnik mass* is defined as:

$$m_B(\Omega, g_0) := \inf\{m_{ADM}(g) : (\bar{M}, g) \in \mathcal{M}(\Omega, g_0)\}$$

where $\mathcal{M}(\Omega, g_0)$ is the space of all asymptotically flat 3-manifolds $(\bar{M}, g)$ satisfying:

- $R_g \geq 0$
- (Ω, g_0) lies in $(\bar{M}, g)$ as an isometric subset.
- $(\bar{M}, g)$ has no horizons. (a non-degeneracy condition preventing $m_{ADM}(g)$ to be arbitrarily small)



The extension (M, g) need to agree with (Ω, g_0) only on the induced metric γ and the mean curvature H on Σ.

Bartnik Minimization Conjecture

Let (Ω, g_0) be a 3-manifold with boundary Σ and nonnegative scalar curvature. The infimum in the definition of the Bartnik mass is realized by an extension (M, g) with boundary $\partial M \cong \Sigma$ that lives in a static vacuum spacetime. More precisely, there exists a metric g and a function f on M satisfying

- The spacetime $(\mathbb{R} \times M, \mathfrak{g}^{(4)} = -f^2 dt^2 + g)$ is Einstein vacuum:

$$\text{Ric}_{\mathfrak{g}} = 0 \quad \text{on } \mathbb{R} \times M \quad \iff \quad \begin{matrix} f\text{Ric}_g = \text{Hess}_g(f) \\ \Delta_g f = 0 \end{matrix} \quad \text{on } M$$

- (M, g) is (AF) and $f = 1$ at infinity.
- $g^T = g_0^T$ and $H_g = H_{g_0}$ on Σ

We say that a pair (g, f) satisfying the above is a *static vacuum extension* realizing the *Bartnik data* (g_0^T, H_{g_0}).

Bartnik Static Extension Conjecture

Given Bartnik boundary data (γ, H) on a compact surface Σ with $H > 0$, there exists a unique static vacuum extension (M, g, f) with $\partial M \cong \Sigma$ realizing the Bartnik boundary data.

Summary of Results

1. We introduce a new framework to tackle extension problems.
   - We write the metric in a geodesic gauge and reduce the system to an elliptic PDE coupled with transport equations.
   - We deal with apparent obstructions fundamental to this problem.
   - We study a nonlocal elliptic system that arises from the linearized problem in the geodesic gauge.
2. We prove local existence and uniqueness of static vacuum extensions for Bartnik data close to that of *any* Schwarzschild sphere.

Main Theorem

Let $m_0 > 0$ and $r_0 > 2m_0$. Let $M = \mathbb{R}^3 \setminus B_{r_0}$. Let $(\mathfrak{g}_{sc}, f_{sc})$ be the Riemannian Schwarzschild extension on M with mass m_0.

- For any (γ, H) close to $({\mathfrak{g}_{sc}}^T, H_{\mathfrak{g}_{sc}})$, in a Sobolev space, on ∂M, there exists a static vacuum extension $(\mathfrak{g}, f)$ on M realizing the Bartnik boundary data $(\mathfrak{g}^T, H_{\mathfrak{g}}) = (\gamma, H)$ on Σ.
- The extension $(\mathfrak{g}, f)$ is "geometrically" unique in a neighbourhood of $(\mathfrak{g}_{sc}, f_{sc})$ in an appropriately defined weighted Sobolev space.

3. We establish solvability of elliptic BVP in the following spaces:

$$u \in \mathcal{A}_{\mathcal{H}_\delta}^{2,k}(M) \iff \begin{matrix} u \in L^2_\delta\left([1,\infty); H^k(S^2)\right), \\ \partial_r u \in L^2_{\delta-1}\left([1,\infty); H^{k-1}(S^2)\right), \\ \partial_r^2 u \in L^2_{\delta-2}\left([1,\infty); H^{k-2}(S^2)\right) \end{matrix}$$

$$u \in \mathcal{A}_{\mathcal{C}_\delta}^{2,k}(M) \iff \begin{matrix} u \in C^0_\delta\left([1,\infty); H^k(S^2)\right), \\ \partial_r u \in C^0_{\delta-1}\left([1,\infty); H^{k-1}(S^2)\right), \\ \partial_r^2 u \in C^0_{\delta-2}\left([1,\infty); H^{k-2}(S^2)\right) \end{matrix}$$

The space of functions we use:

$$\mathcal{A}_\delta^{2,k}(M) := \mathcal{A}_{\mathcal{H}_\delta}^{2,k}(M) \bigcap \mathcal{A}_{\mathcal{C}_\delta}^{2,k}(M)$$

Solvability of Elliptic PDEs in Non-traditional Spaces

Fix $\delta \in (-1, -\frac{1}{2})$. Let g be an asymptotically flat metric on $M = [1, \infty) \times S^2$. Define the operator Q by $Q(u) = (\Delta_g u, u|_{\partial M})$. Then

$$Q : \mathcal{A}_{\mathcal{H}_\delta}^{2,k}(M) \to L^2_{\delta-2}\left([1,\infty); H^{k-2}(S^2)\right) \times H^{k-1/2}(S^2) \text{ is an isomorphism}$$

$$Q : \mathcal{A}_{\mathcal{C}_\delta}^{2,k}(M) \to C^0_{\delta-2}\left([1,\infty); H^{k-2}(S^2)\right) \times H^{k}(S^2) \text{ is an isomorphism}$$

The Definition of the Modified Problem

Define the map Φ by:

$$\Phi(\gamma_{\mathfrak{B}}, \frac{1}{2}trK_{\mathfrak{B}}, g, u, X) := \begin{pmatrix} \Delta_g u \\ \partial_r trK + \frac{1}{2}trK^2 + |\hat{K}|^2 + 2(\partial_r u)^2 \\ \nabla_{\frac{\partial}{\partial r}} \hat{K} + trK\,\hat{K} + \left[2\not{d}u \otimes \not{d}u + g(r)\left((\partial_r u)^2 - |\nabla u|^2\right)\right] \\ \Delta_{g,conf}(F(X)X) \\ 2|\not{\nabla} u|^2 - 2(\partial_r u)^2 - |\hat{K}|^2 - R_{\partial M} + \frac{1}{2}trK^2 \\ 2(\partial_r u)\not{d}u - \not{div}(\hat{K}) + \frac{1}{2}\not{d}trK + \omega(g, X_\infty) \\ e^{-2u} g|_{\partial M} - \gamma_{\mathfrak{B}} \\ trK|_{\partial M} - e^{-u}\left(trK_{\mathfrak{B}} + 2e^u \partial_r u\right) \end{pmatrix}$$

$$X := X_0 + X_\infty \in \widetilde{\mathcal{X}}^2_\delta \oplus \mathcal{X}_\infty, \quad \Delta_{g,conf} X := \text{div}_g\left(\widehat{\mathcal{L}_X g}\right), \quad \omega(g, X_\infty) := (X_\infty|_{\partial M})^\flat$$

X is an artificial vector field we add to the definition of the solution to circumvent a 6-dim space of apparent obstructions that appear.

Definition of the Modified Solution:
(g, u, X) is a solution to the modified problem with boundary data $(\gamma_{\mathfrak{B}}, \frac{1}{2}trK_{\mathfrak{B}})$ if $\Phi(\gamma_{\mathfrak{B}}, \frac{1}{2}trK_{\mathfrak{B}}, g, u, X) = 0$.
Furthermore, (g, u) is a static vacuum extension if $X = 0$.

Vanishing of the Artificial Vector field

In fact, if (g, u, X) is a modified solution, then $F(X)X$ is conformal Killing on (M, g) that vanishes at infinity and so vanishes everywhere by the following lemma.

Non-existence of Nontrivial Conformal Killing Fields Vanishing at Infinity

Consider the metric $dr^2 + r^2(\gamma_\infty + h(r))$ on M where, for some $\delta < 0$,

$$\gamma_\infty \in \mathcal{M}^k(S^2) \text{ and } h \in H^2_\delta\left([nm_0, \infty); \mathcal{H}^k(S^2)\right)$$

Suppose that γ_∞ is close to γ_{S^2} in the H^k norm. Then there is no nontrivial conformal Killing vector fields vanishing at infinity.

The Linearized Problem

$$D\Phi_{sc}(\tilde{g}, \tilde{u}, \tilde{X}) = \begin{pmatrix} \Delta_{g_{sc}}\tilde{u} + (\partial_r u_{sc})(\widetilde{trK}) \\ \partial_r \widetilde{trK} + trK_{sc}\widetilde{trK} + 4(\partial_r u_{sc})(\partial_r \tilde{u}) \\ \mathcal{L}_{\frac{\partial}{\partial r}} \widetilde{\hat{K}} \\ \Delta_{g_{sc},conf}\tilde{X} \\ -4(\partial_r u_{sc})(\partial_r \tilde{u}) + trK_{sc}\,\widetilde{trK}\Big|_{\partial M} + \frac{4}{n(n-2)m_0^2}\tilde{u} + 2\not{\Delta}_{\gamma_{sc}}\tilde{u} \\ 2(\partial_r u_{sc})\not{d}\tilde{u} - \not{div}(\widetilde{\hat{K}}) + \tilde{\omega} \\ \frac{n}{n-2}\tilde{\gamma} - 2n^2 m_0^2\,\tilde{u}\,g_{S^2} \\ \widetilde{trK}\Big|_{\partial M} + \frac{2}{nm_0}\tilde{u} - 2\partial_r \tilde{u} \end{pmatrix}$$

The equations decouple to give a non-local elliptic system on $\tilde{u}$:

$$\begin{cases} \Delta_{g_{sc}}\tilde{u} - \frac{4m_0^2}{[r(r-2m_0)]^2}\tilde{u} + \frac{n(n-2)m_0}{2}\frac{4m_0^2}{[r(r-2m_0)]^2}\left(\frac{4-n}{n(n-2)m_0}\tilde{u}|_{\partial M} + \partial_r \tilde{u}|_{\partial M}\right) = \psi \\ \frac{2}{nm_0}\partial_r \tilde{u} + \not{\Delta}\tilde{u} + \frac{2}{n^2(n-2)m_0^2}\tilde{u} = \Gamma \end{cases}$$

Showing that $D\Phi_{sc}$ is an isomorphism reduces to the solving the above system!

The Non-local Elliptic System

Solving the linearized problem reduces to showing that the following nonlocal elliptic operator is an isomorphism:

$$\mathcal{P}_{sc} : \mathcal{A}_\delta^{(2,k+1)}(M) \to \mathcal{A}_{\delta-2}^{(0,k-1)}(M) \times H^{k-1}(\partial M)$$

$$\mathcal{P}_{sc}(\tilde{u}) := \begin{pmatrix} \Delta_{g_{sc}}\tilde{u} - \frac{4m_0^2}{[r(r-2m_0)]^2}\tilde{u} + \frac{n(n-2)m_0}{2}\frac{4m_0^2}{[r(r-2m_0)]^2}\left(\frac{4-n}{n(n-2)m_0}\tilde{u}|_{\partial M} + \partial_r \tilde{u}|_{\partial M}\right) \\ \frac{2}{nm_0}\partial_r \tilde{u} + \not{\Delta}\tilde{u} + \frac{2}{n^2(n-2)m_0^2}\tilde{u} \end{pmatrix}$$

Outline of the proof:

1. The operator $Q(\tilde{u}) = (\Delta_{sc}\tilde{u}, \tilde{u}|_{\partial M})$ is an isomorphism from $\mathcal{A}_\delta^{(2,k+1)}(M)$ to $\mathcal{A}_{\delta-2}^{(0,k-1)}(M) \times H^{k-1}(\partial M)$.
2. $\mathcal{P}_{sc}$ is a compact perturbation of the operator $\tilde{u} \mapsto (\Delta_{sc}\tilde{u}, \not{\Delta}\tilde{u})$, and so is Fredholm of index 0.
3. Kernel of $\mathcal{P}_{sc}$ is trivial.

The above hinges on the following new uniform estimates for the Legendre functions of the first and second kind: There exists a constant $C = C(R)$ such that for any $\ell \geq 1$ and $z \in [R, \infty)$, the following holds

$$z^{-\ell}|P_\ell(z)| \leq C\left(\frac{2z}{z + \sqrt{z^2-1}}\right)^{-\ell}, \quad z^{\ell+1}|Q_\ell(z)| \leq C\left(\frac{2z}{z + \sqrt{z^2-1}}\right)^{\ell}$$