Photon spheres from a perturbative perspective

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Setup

Studying the Schwarzschild solution to the Einstein equations, a feature that appears almost as natural as the existence of an event horizon, is the presence of a photon sphere, P^n, at $r = 3m$. This hypersurface does not only appear in the Schwarzschild solution, but also in many other space-times. We will, however, focus only on the class of spherically symmetric solutions, that is the space-times which metric can be expressed as follows:

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\mu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (1)$$

Photon surface and Photon sphere

Definition (photon surface): Let us consider a lorentzian space-time $(\mathcal{L}^{n+1}, \mathfrak{g})$. We define a **photon surface** to be a time-like hypersurface immersed in $(\mathcal{L}^{n+1}, \mathfrak{g})$ such that every null-geodesic initially tangent to P^n remains tangent at all times.

Definition (photon sphere): if we additionally require $(\mathcal{L}^{n+1}, \mathfrak{g})$ to be static and that each connected component of the photon surface has constant lapse function, then we call this photon surface a **photon sphere**.

Let us call p the metric induced on this hypersurface and χ its second fundamental form. Then, Theorem 2.2 in [1] gives operative conditions for (P^n, p) being a photon surface. In particular one of them is that (P^n, p) is a photon surface if, and only if, it is a totally umbilic hypersurface:

$$\chi_{\mu\nu} = \frac{tr_p\chi}{n}\, p_{\mu\nu}. \quad (2)$$

Construction of the perturbation

Consider a metric of the form as the one given in (1) and assume $\exp\{(2\nu)\} = (1 - 2m/r) = \exp\{(-2\mu)\}$. We can then perturb this metric to

$$g = g^{(0)} + \varepsilon g^{(1)} + \varepsilon^2 g^{(2)} + \ldots$$

where $g^{(0)}$ is the background metric. For simplicity let us consider only the first order perturbation, since, from section 3 of [2], the higher order terms can be treated in a similar manner. Let $h_{\mu\nu} = \varepsilon g^{(1)}_{\mu\nu}$ be a small variation, then we can write the linearization of the Einstein equations as

$$R_{\mu\nu} + \delta R_{\mu\nu} = 0. \quad (3)$$

Because the background metric is Schwarzschild the first term vanish, then $\delta R_{\mu\nu} = 0$ implies that the perturbed space-time is also in vacuum.

This are hyperbolic, non linear, partial differential equations which we hope to solve and obtain a perturbation that can be separated in terms that depend only on one of the components. The tool that does the trick is the development in spherical harmonics. From this we obtain two possible solutions for the perturbation one for odd waves and the other for even ones.

Using the change of gauge in [3] it is possible to simplify the form of the perturbation so that it only depends on the radial coordinate. In the specific case of the even parity solution with $m = 0,\ \ell > 2$ (where m and ℓ are the coefficients of the spherical harmonics) the solution to the PDE obtained by plugging the variation in the first variation of the Einstein equations is

$$H(r) = \alpha P^{(2)}_\ell(1 - \frac{r}{m}) + \beta Q^{(2)}_\ell(1 - \frac{r}{m}) \quad (4)$$

Physical interpretation of the terms

Comparing the result to the case of the electrostatic potential, it is possible to give a possible interpretation to the two terms.

The first, $\alpha P^{(2)}_\ell(1 - r/m)$, can be thought to be the variation of the spherical symmetry due to "internal masses", while $\beta Q^{(2)}_\ell(1 - r/m)$ can be thought of as a perturbation due to "masses at infinity". Notice, however, that by assumption we are working in vacuum, and that the actual meaning of "internal mass" or "mass at infinity" is therefore different from the one we usually mean.

Considerations on the uniqueness

Before going into the proof of uniqueness, it is important to understand the following. Even if we can rule out the existence of regular solutions branching from Schwarzschild, or any other spherically symmetric space-time, it is technically possible for a sequence of regular solutions to approach, in the limit, a spherically symmetric space-time in a singular way. Therefore, proving the uniqueness by showing that there exists not perturbative solutions is not enough.

At the same time proving that a first order perturbation exists does not mean that we managed to construct a non linear solution, This is because it might be that the second or third order perturbative term is not compatible with the photon surface condition.

Perturbing the spherical symmetry on time slices

In [4] C. Cederbaum proved that the existence of a photon sphere around a static, asymptotically flat space-time in $\mathcal{S}$ (that is the class of space-times with a spherically symmetric metric), assuming a constant lapse function, implies that the space-time in question must be Schwarzschild. Physically this assumption is justified by saying that the photon sphere is monochromatic, since by definition $g(\dot\gamma, N^{-1}\partial_t) = E = \nu\hbar$.

In [2] Yoshino takes an orthogonal direction. He sets the lapse function to be a non-constant function in the spacial components, maintaining it static in the time evolution. The idea is that we want to check if, perturbing the spacial spherical symmetry, there exists some other metric that allows for the existence of a photon sphere.

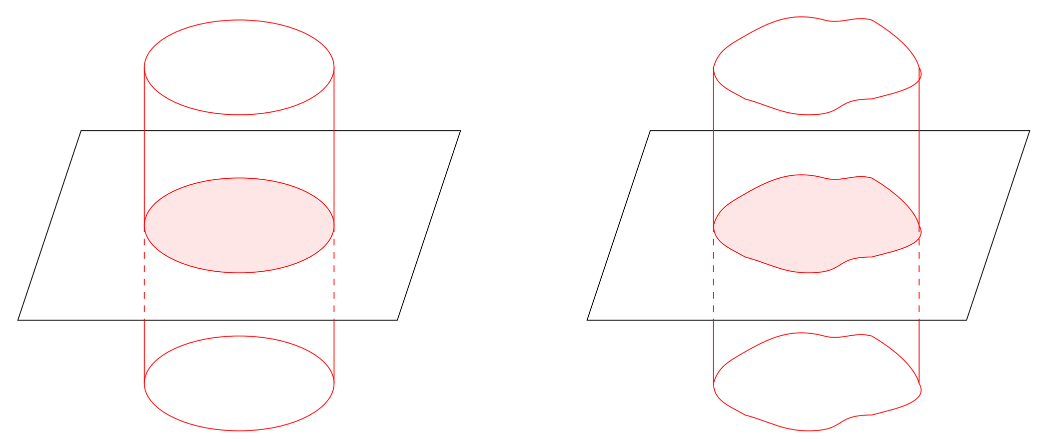

Figure 1. The right picture shows a space-time as it appears in the hypothesis of [4], which evolves as a cylinder. On the left a possible space-time that has non-constant static lapse function.

If so, the photon sphere in distorted Schwarzschild will be found at a radius $r = f(\theta, \varphi)$ where $f = f^{(0)} + \varepsilon f^{(1)} + \ldots$ with $f^{(0)} = 3m$. Then the perturbed photon sphere condition becomes

$$f^{(1)}_{,\theta\varphi} = \cot\theta\, f^{(1)}_{,\theta}; \quad (5)$$

$$f^{(1)}_{,\varphi\varphi} = \sin^2\theta\, f^{(1)}_{,\theta\theta} - \sin\theta\cos\theta\, f^{(1)}_{,\theta}; \quad (6)$$

$$\left(\nu^{(1)}_{,r} - \psi^{(1)}_{,r}\right)\Big|_{r=3m} = \frac{1}{3m}\left(f^{(1)} - f^{(1)}_{,\theta\theta}\right). \quad (7)$$

The solution of the first two equations is a linear composition of the spherical harmonics with $(m, \ell) = (0,0), (1,\pm1)$, and $(1, 0)$. For the last one, we notice that the left hand side has modes $\ell = 0, 1$ while the right hand side has $\ell \geq 2$. This implies that $f^{(1)} = 0$ and the condition for $r = 3m$ to be a photon sphere becomes

$$\left(\nu^{(1)}_{,r} - \psi^{(1)}_{,r}\right)\Big|_{r=3m} = 0, \quad (8)$$

where $\psi^{(1)}$ is the the perturbation of the exponent in the exponential term in front of the angular component of the metric ($\exp\{2\psi\}$ with $\psi^{(0)} = 0$). Consider now the region outside of $r = 3m$ and assume "vacuum", i.e. $\alpha = 0$. Combining equation (4) and (8) we get

$$\frac{H^{(1)}_{,x}}{H^{(1)}}\Bigg|_{x=2} = \frac{1}{3}$$

where $x = r/m - 1$. Which is not satisfied in this setting. Therefore, if we perturb a metric around Schwarzschild using a static perturbation, no photon sphere is allowed.

What's next?

Some progress has been done on the uniqueness of the photon sphere in the class of spherically symmetric, asymptotically flat space-times $\mathcal{S}$ from the publication of Yoshino's paper. In [5], for example, C. Cederbaum and G. Galloway showed that in this setting, relaxing the assumption of constant lapse function by making it equipotential, uniqueness holds still.

The goal for the master thesis is to generalize the computations done in [3] by Regge and Wheeler to the whole class $\mathcal{S}$. And then use the method employed by Yoshino to prove uniqueness by relaxing the staticity assumption on the lapse function, replacing it with the equipotential assumption. This can be physically interpreted as having a non monochromatic photon sphere.

References

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