Some ongoing Efforts for Evolving Einstein Field Equations on Hyperboloidal Slices

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Motivation

- Why Hyperboloidal slices?
  - Need to include future null infinity, $\mathscr{I}^+$, in the computational domain with a well-posed formalism.
- Why $\mathscr{I}^+$?
  - The only region where Gravitational Waves (GW) can be defined unambiguously.
  - Useful for studying phenomena of fundamental interest, like weak cosmic censorship conjecture, black hole formation, etc.
- Existing methods like Extrapolation from Cauchy evolution, CCE and CCM introduce errors due to their limitations.

Existing method 1: Extrapolation

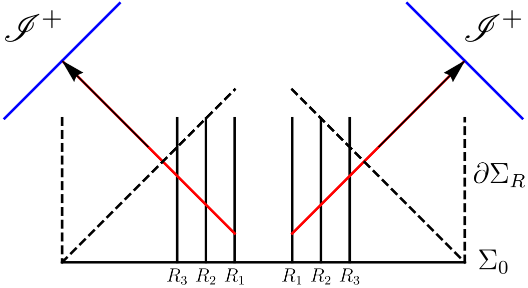


Figure: Extrapolation schematics.

Extrapolation Methods from Cauchy evolution:

- Boundary Errors, Extrapolation Errors.

Existing method 2: Cauchy Characteristic Extraction (CCE): (PRD 54, 6153)

- Data extracted at a timelike artificial boundary at a finite radius. Introduces gauge ambiguities.
- Unphysical boundary conditions at the boundary of the Cauchy domain. Limits time evolution.
- One-sided propagation of the signal.

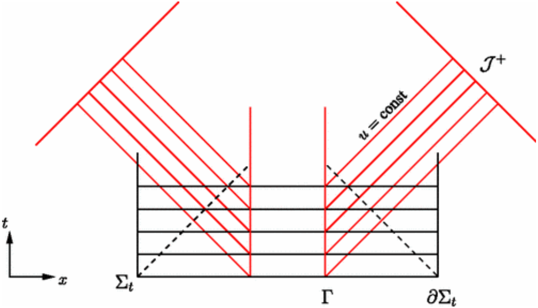


Figure: CCE schematics.

Source: C. Reisswig, N. T. Bishop, D. Pollney, and B. Szilgyi, Phys. Rev. Lett. 103, 221101 (2009).

Proposed method: Cauchy Characteristic Matching (CCM): (Winicour, 2009).

- Cauchy and Characteristic regions evolved at once. Data exchanged through boundary.
- 3D, in construction.
- Present formulations possibly ill-posed. (e.g. PRD 102, 064035).

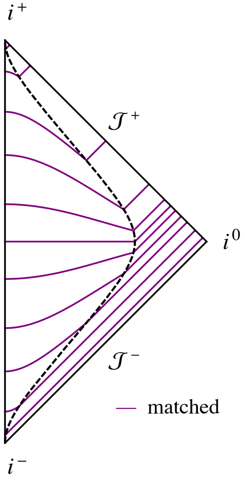


Figure: CCM schematics. Credit: Alex Vañó-Viñuales.

Hyperboloidal Slices[1],[2]

- Meet $\mathscr{I}^+$, instead of i^0 $\Rightarrow$ signal can be directly collected at $\mathscr{I}^+$.
- Overcomes all problem encountered in the existing methods.
- More gauge freedom than null slices.
- Suitable for dealing with initial-boundary value problems when initial data is not of compact support.

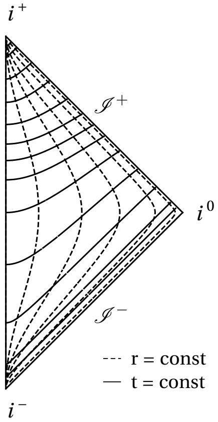


Figure: Hyperboloidal slices in Minkowski spacetime.

[1]Mathematical front: Friedrich 1981-86, LeFloch et. al. 2014-19 etc.
[2]Numerical front: Hübner 1993, Frauendiener 1998-2006, Zenginoğlu 2005-11, Moncrief and Rinne 2009-18, Vañó-Viñuales 2015-18 etc.

Formulating EFEs on Hyperboloidal Slices

ToDo list:

- Goal 1: Derive a well-posed formulation of the Einstein Field Equations (EFEs) on hyperboloidal slices.
- Goal 2: Derive a discretization scheme preserving the well-posedness for the discrete Einstein Field Equation (dEFE) on hyperboloidal slices.

Our Approach:

- Goal 1a: Achieved by using the Dual-Foliation (DF) formalism[3] together with the Generalized Harmonic Gauge (GHG)[4].
- Goal 1b: Derive a suitable regularization scheme.
- Goal 2: Discretization scheme: Need to start from scratch, the Linear Wave Equation.

[3](Hilditch 2015, 2016)

[4]Fourès-Bruhat 1952, Lindblad and Rodnianski 2003, Lindblom 2006, Gasperín and Hilditch 2018 etc.

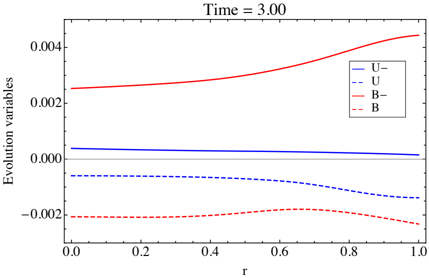
Vacuum EFEs: Asymptotic Analysis (GBU Paper)[5]

E. Gasperin, SG, D. Hilditch, A. Vañó-Viñuales, CQG, Vol. 37, No. 3

- EFEs in Harmonic Gauge (HG) give three classes of asymptotic equations.
- Good fields 'g', Bad fields 'b', Ugly fields 'u':

$$\Box g = 0\,, \quad \Box b = (\partial_T g)^2\,,$$
$$\Box u = \frac{2}{R}\partial_T u\,.$$

- $g \sim 1/R$, $b \sim \ln R/R$, $u \sim 1/R^2$ towards $\mathscr{I}^+$.



[5]See also: Gasperín and Hilditch, 2019, CQG 36, 195016.

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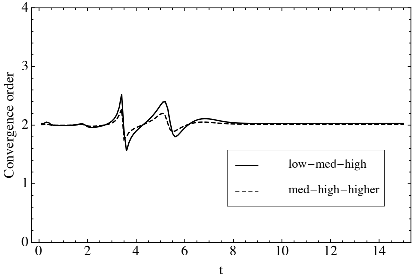

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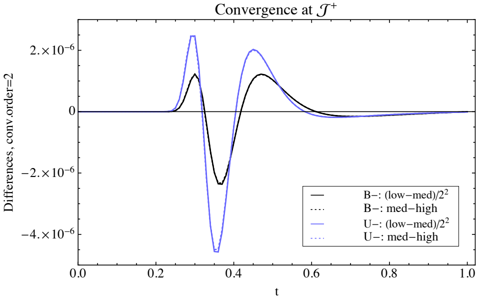

Goal 2: Summation-by-Parts (SBP) scheme

SG, A. Vañó-Viñuales, D. Hilditch, S. Bose, PRD 103, 084045

- Derived for the class of equations

$$(\Box - F)\psi = 0\,,$$

  all in spherical symmetry, $F = F(R)$, $\psi = \psi(T, R)$. (one e.g. in Ma's talk)

- Assures stability by guaranteeing $\dot{\hat{E}} \leq 0$ on these slices, for all times.

- Lax Equivalence Theorem implies convergence at the desired order.

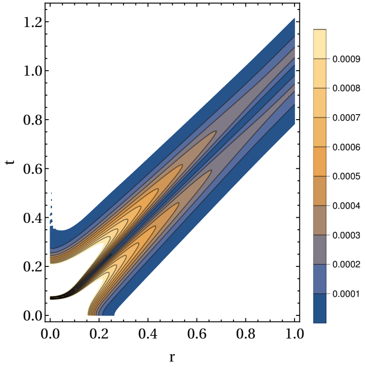

Figure: Propagation of a narrow pulse to $\mathscr{I}^+$.

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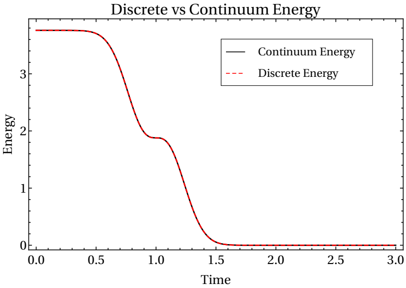

Figure: Discrete energy vs continuum energy.

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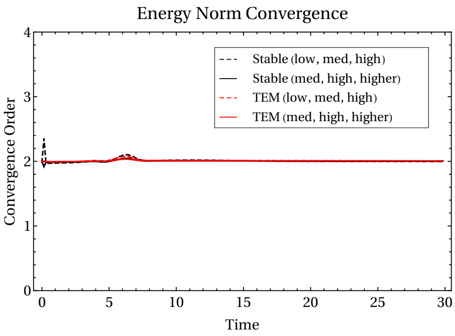

Figure: Convergence order in the energy norm.

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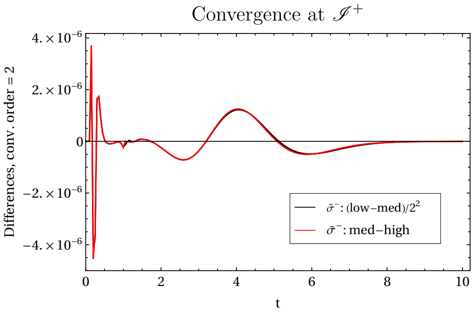


Figure: Convergence order at $\mathscr{I}^+$.

Spherical GR: (Ongoing)

- Why Spherical symmetry?
  - To solve the problems involving spherical symmetry, like spherical collapse etc.
  - A stepping stone to full 3d.

- In spherical symmetry, the metric has the following general form

$$\mathbf{g} = \begin{pmatrix} g_{TT} & g_{TR} & 0 & 0 \\ g_{TR} & g_{RR} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ 0 & 0 & 0 & g_{\theta\theta}\sin^2\theta \end{pmatrix}.$$

Spherical GR: (Ongoing)

- Converting into a GBU structure requires the following form

$$\mathbf{g} = \begin{pmatrix} \frac{2e^{\delta} C_+ C_-}{C_+ - C_-} & \frac{e^{\delta}(C_- + C_+)}{C_- - C_+} & 0 & 0 \\ \frac{e^{\delta}(C_- + C_+)}{C_- - C_+} & \frac{2e^{\delta}}{C_+ - C_-} & 0 & 0 \\ 0 & 0 & \mathring{R}^2 & 0 \\ 0 & 0 & 0 & \mathring{R}^2 \sin^2\theta \end{pmatrix} .$$

- Dynamical variables: $C_{\pm}$, δ, $\mathring{R}$, all functions of (T, R).

Spherical GR:

For your eyes only!

- ‘Causal structure’ variables $C^R_\pm$:

$$\nabla_\xi \left(\frac{\mathring{R}^2}{e^\delta \kappa} \nabla_{\underline{\xi}} C^R_+ \right) + \mathring{R} \nabla_\xi H^\eta + \mathring{R} (\nabla_\xi \mathring{R}) \nabla_\xi C^R_+ + 8\pi e^{-\delta} \mathring{R}^2 T_{\xi\xi} = 0\,,$$

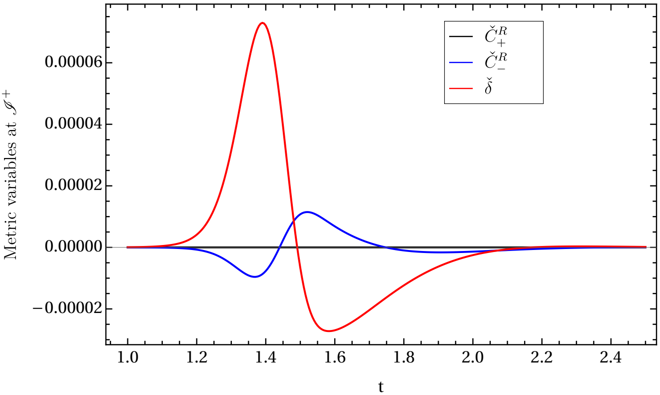
$$\nabla_{\underline{\xi}} \left(\frac{\mathring{R}^2}{e^\delta \kappa} \nabla_\xi C^R_- \right) + \mathring{R} \nabla_{\underline{\xi}} H^{\underline{\eta}} + \mathring{R} (\nabla_{\underline{\xi}} \mathring{R}) \nabla_{\underline{\xi}} C^R_- + 8\pi e^{-\delta} \mathring{R}^2 T_{\underline{\xi}\underline{\xi}} = 0\,.$$

- ‘Determinant’ variables $(e^\delta, \mathring{R})$:

$$\nabla_a \left(\frac{\xi^a}{e^\delta \kappa} \nabla_{\underline{\xi}} \delta \right) - \mathring{R} \nabla_a \left(\frac{H^a}{\mathring{R}} \right) + \frac{2}{\mathring{R}^2 e^\delta \kappa} (\nabla_\xi \mathring{R})(\nabla_{\underline{\xi}} \mathring{R})$$
$$+ \frac{1}{4 e^\delta \kappa^3} \left(\nabla_{\underline{\xi}} C^R_- \nabla_\xi C^R_+ - \nabla_{\underline{\xi}} C^R_+ \nabla_\xi C^R_- \right) + 16\pi T_T = 0\,,$$

$$\nabla_a \left(\frac{\xi^a}{e^\delta \kappa} \nabla_{\underline{\xi}} \mathring{R}^2 \right) + 2 = 0\,.$$

Some preliminary results:



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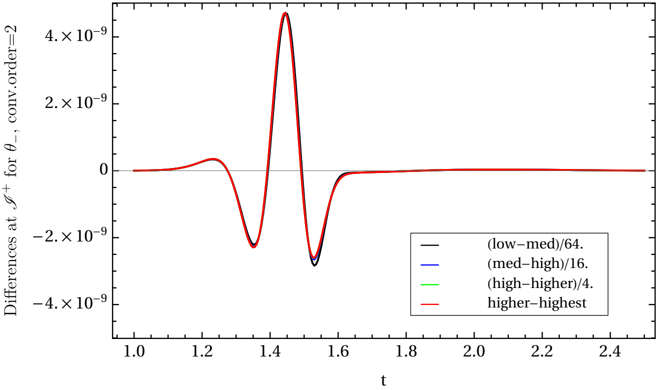

Summary and Conclusions

- Continuum analysis:
  - Using asymptotic analysis, derived a regularization scheme on hyperboloidal slices for a 'model system' of equations that mimic the asymptotic form of the Einstein Field Equations (EFEs) in harmonic gauge (HG).
  - Propose that the EFEs in HG, and in generalized harmonic gauge (GHG) can be regularized similarly (work in progress).
- Discrete analysis:
  - Derived a 'good' discretization scheme on hyperboloidal slices that assures stability, and hence convergence, for a class of linear wave equations.
  - Propose that a similar scheme can improve the numerical results for the EFEs in GHG (future plan).
- Obtained some promising results for the spherical EFEs in GHG on hyperboloidal slices (stay tuned).

Thank You for your Attention!!
Questions? Comments?

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Dual-Foliation Formalism

- Consider two foliations of a spacetime $X^{\underline{\mu}} = (T, X^{\underline{i}})$ and $x^{\mu} = (t, x^{i})$, in spacelike hypersurfaces and relate the two geometries.

- Using a $3+1$ split of the Jacobian $J^{\underline{\mu}}{}_{\mu} = dX^{\underline{\mu}}/dx^{\mu}$

$$\partial_T = (J^{-1})^{\mu}{}_{\underline{T}}\,\partial_{\mu}\,, \quad \partial_{\underline{i}} = (J^{-1})^{\mu}{}_{\underline{i}}\,\partial_{\mu}\,.$$

- Dynamical variables, or the tensor basis, remain the same. Only the coordinates are transformed.

- An equation for a state vector $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ of the form

$$\partial_T \mathbf{u} = \mathbf{A}^{\underline{p}}\,\partial_{\underline{p}}\mathbf{u} + A\,\mathbf{S}\,,$$

becomes

$$\partial_t \mathbf{u} = \mathbf{B}^{p}\,\partial_{p}\mathbf{u} + B\,\mathbf{S}\,.$$

Nonlinear Change of Variables

- Define Operator

$$L^a = (\partial_T)^a + (\partial_R)^a \,, \qquad \underline{L}^a = (\partial_T)^a - (\partial_R)^a \,, \quad s = \ln R \,.$$

- Good Fields:

$$G^+ = RL(Rg) \,, \quad G^- = R\underline{L}g \,, \quad G = Rg \,.$$

- Bad Fields:

$$B^+ = RL(Rb + \frac{1}{8}s\eta) \,, \quad B^- = R\underline{L}b + \frac{1}{8}s\underline{L}\eta \,, \quad B = Rb + \frac{1}{8}s\eta \,.$$

- Ugly Fields:

$$U^+ = RL(R^2 u) \,, \quad U^- = R^2\underline{L}u \,, \quad U = R^2 u \,.$$

- $\partial_T \eta = (G^-)^2$.