The Maxwell-scalar field system near spatial infinity

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Outline

- The cylinder at spatial infinity and the F-gauge
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  - The F-gauge and the conformal structure.

- The Maxwell-scalar field system
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The cylinder at spatial infinity and the F-gauge

The standard representation of spatial infinity

- The Minkowski metric $\tilde{\boldsymbol{\eta}}$ written in Cartesian coordinates $(\tilde{x}^\mu) = (\tilde{t}, \tilde{x}^\alpha)$,

$$\tilde{\boldsymbol{\eta}} = \eta_{\mu\nu}\mathbf{d}\tilde{x}^\mu \otimes \mathbf{d}\tilde{x}^\nu,$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. By introducing polar coordinates defined by $\tilde{\rho}^2 \equiv \delta_{\alpha\beta}\tilde{x}^\alpha\tilde{x}^\beta$ where $\delta_{\alpha\beta} = \mathrm{diag}(1,1,1)$, and an arbitrary choice of coordinates on $\mathbb{S}^2$, the metric $\tilde{\boldsymbol{\eta}}$ can be written as

$$\tilde{\boldsymbol{\eta}} = \mathbf{d}\tilde{t} \otimes \mathbf{d}\tilde{t} - \mathbf{d}\tilde{\rho} \otimes \mathbf{d}\tilde{\rho} - \tilde{\rho}^2\boldsymbol{\sigma},$$

with $\tilde{t} \in (-\infty, \infty)$, $\tilde{\rho} \in [0, \infty)$ and where $\boldsymbol{\sigma}$ denotes the standard metric on $\mathbb{S}^2$. A conformal representation of the Minkowski spacetime close to i^0 is to make use of *inversion coordinates* $(x^\alpha) = (t, x^i)$ defined by

$$x^\mu = -\tilde{x}^\mu/\tilde{X}^2, \qquad \tilde{X}^2 \equiv \tilde{\eta}_{\mu\nu}\tilde{x}^\mu\tilde{x}^\nu,$$

which is valid in the domain

$$\tilde{\mathcal{D}} \equiv \{p \in \mathbb{R}^4 \mid \eta_{\mu\nu}\tilde{x}^\mu(p)\tilde{x}^\nu(p) < 0\}.$$

Using these coordinates the conformal representation of the Minkowski spacetime with *unphysical metric* is given by

$$\bar{\boldsymbol{\eta}} = \Xi^2\tilde{\boldsymbol{\eta}}, \qquad \Xi \equiv X^2.$$

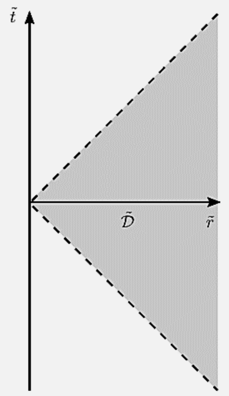
Introducing an *unphysical polar coordinate* via the relation $\rho^2 \equiv \delta_{\alpha\beta}x^\alpha x^\beta$, one finds that the metric $\bar{\boldsymbol{\eta}}$ can be written as

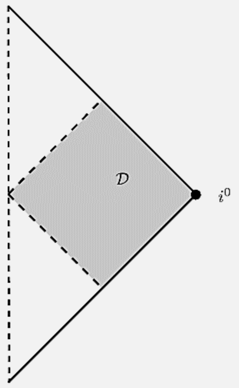
$$\bar{\boldsymbol{\eta}} = \mathbf{d}t \otimes \mathbf{d}t - \mathbf{d}\rho \otimes \mathbf{d}\rho - \rho^2\boldsymbol{\sigma}, \qquad \Xi = t^2 - \rho^2,$$

with $t \in (-\infty, \infty)$ and $\rho \in (0, \infty)$. In this conformal representation, spatial infinity i^0 corresponds to the origin of the domain

$$\mathcal{D} \equiv \{p \in \mathbb{R}^4 \mid \eta_{\mu\nu}x^\mu(p)x^\nu(p) < 0\}.$$

This region contains the asymptotic region of the Minkowski spacetime around spatial infinity.





The F-gauge and the conformal structure

The F-gauge

Introducing a time coordinate τ through the relation $t = \rho\tau$ one finds that the metric $\bar{\boldsymbol{\eta}}$ can be written as

$$\bar{\boldsymbol{\eta}} = \rho^2 \mathbf{d}\tau \otimes \mathbf{d}\tau - (1-\tau^2)\mathbf{d}\rho \otimes \mathbf{d}\rho + \rho\tau \mathbf{d}\rho \otimes \mathbf{d}\tau + \rho\tau \mathbf{d}\tau \otimes \mathbf{d}\rho - \rho^2 \boldsymbol{\sigma}.$$

The conformal representation containing the *cylinder at spatial infinity* is obtained by considering the rescaled metric

$$\boldsymbol{\eta} \equiv \frac{1}{\rho^2}\bar{\boldsymbol{\eta}}.$$

Introducing the coordinate $\varrho \equiv -\ln \rho$ the metric $\boldsymbol{\eta}$ can be reexpressed as

$$\boldsymbol{\eta} = \mathbf{d}\tau \otimes \mathbf{d}\tau - (1-\tau^2)\mathbf{d}\varrho \otimes \mathbf{d}\varrho - \tau \mathbf{d}\tau \otimes \mathbf{d}\varrho - \tau \mathbf{d}\varrho \otimes \mathbf{d}\tau - \boldsymbol{\sigma}.$$

The conformal structure

Given the conformal extension $(\mathcal{M}, \boldsymbol{\eta})$ where

$$\boldsymbol{\eta} = \Theta^2 \tilde{\boldsymbol{\eta}}, \qquad \Theta \equiv \rho(1-\tau^2),$$

and

$$\mathcal{M} \equiv \{p \in \mathbb{R}^4 \mid -1 \le \tau \le 1, \;\; \rho(p) \ge 0\}.$$

In this representation future and past null infinity are described by the sets

$$\mathscr{I}^+ \equiv \{p \in \mathcal{M} \mid \tau(p) = 1\}, \qquad \mathscr{I}^- \equiv \{p \in \mathcal{M} \mid \tau(p) = -1\},$$

while the physical Minkowski spacetime can be identified with the set

$$\tilde{\mathcal{M}} \equiv \{p \in \mathcal{M} \mid -1 < \tau(p) < 1, \;\; \rho(p) > 0\},$$

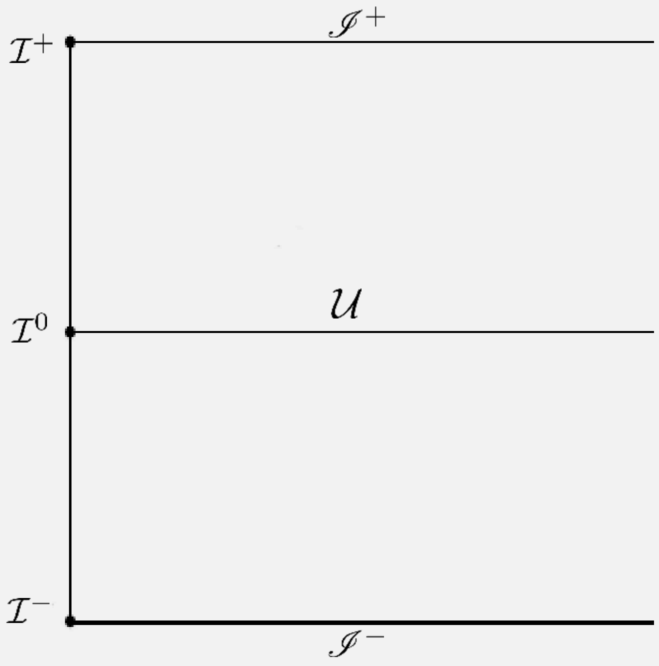
In addition, the *cylinder at spatial infinity* is given by

$$\mathcal{I} \equiv \{p \in \mathcal{M} \mid \; |\tau(p)| < 1, \; \rho(p) = 0\},$$

and *critical sets* are

$$\mathcal{I}^+ \equiv \{p \in \mathcal{M} \mid \tau(p) = 1, \; \rho(p) = 0\}, \qquad \mathcal{I}^- \equiv \{p \in \mathcal{M} \mid \tau(p) = -1, \; \rho(p) = 0\}.$$

Spatial infinity i^0, which is at infinity respect to the metric $\boldsymbol{\eta}$, corresponds to a set which has the topology of $\mathbb{R} \times \mathbb{S}^2$.



The Maxwell-scalar field system

Conformal properties

Let $\tilde{F}_{ab}$ denote the *the Faraday tensor* over a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and let $\tilde{\nabla}$ be the Levi-Civita connection of the metric $\tilde{\boldsymbol{g}}$. The Maxwell equations with source are given by

$$\tilde{\nabla}^a \tilde{F}_{ab} = \tilde{J}_b,$$
$$\tilde{\nabla}_{[a}\tilde{F}_{bc]} = 0.$$

The homogeneous equation is automatically satisfied if one sets

$$\tilde{F}_{ab} = \tilde{\nabla}_a \tilde{A}_b - \tilde{\nabla}_b \tilde{A}_a.$$

Coupled to the above, we consider the *conformally invariant wave equation*

$$\tilde{\mathfrak{D}}_a \tilde{\mathfrak{D}}^a \tilde{\phi} - \frac{1}{6}\tilde{R}\tilde{\phi} = 0,$$

The coupling between the Maxwell field and the scalar field is encoded in the covariant derivative

$$\tilde{\mathfrak{D}}_a = \tilde{\nabla}_a - \mathrm{i}\mathfrak{q}\tilde{A}_a.$$

The current $\tilde{J}_a$ is given by

$$\tilde{J}_a = \mathrm{i}\mathfrak{q}\left(\bar{\tilde{\phi}}\tilde{\mathfrak{D}}_a\tilde{\phi} - \tilde{\phi}\overline{(\tilde{\mathfrak{D}}_a\tilde{\phi})}\right)$$

Consider a conformal rescaling of the form

$$g_{ab} = \Xi^2 \tilde{g}_{ab}.$$

Associated to the latter we define the *unphyiscal Faraday tensor*, *unphysical vector potential* and the *unphysical scalar field* via

$$F_{ab} \equiv \tilde{F}_{ab}, \qquad A_a \equiv \tilde{A}_a, \qquad \phi \equiv \Xi^{-1}\tilde{\phi},$$

Spinorial expressions

Let $F_{AA'BB'}$ denote the spinorial counderpart of the Faraday tensor F_{ab}. It satisfies the well-know decomposition

$$F_{AA'BB'} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB},$$

where $\phi_{AB} = \phi_{(AB)}$ is the so-called *Maxwell spinor*. The unphysical Maxwell equations are equivalent to

$$\nabla^{B}{}_{A'}\phi_{AB} = J_{AA'},$$

where

$$J_{AA'} \equiv \mathrm{i}q\left(\bar{\phi}\nabla_{AA'}\phi - \phi\nabla_{AA'}\bar{\phi}\right) + 2\mathfrak{q}^2|\phi|^2 A_{AA'}.$$

The study of the Maxwell-scalar field system can be reduced, making use of the generalised Lorenz gauge condition

$$\nabla^{AA'}A_{AA'} = 0,$$

to the system of wave equations

$$\textbf{(A)} \qquad \Box\phi = q^2\phi A_{AA'}A^{AA'} + 2\mathrm{i}qA^{AA'}\nabla_{AA'}\phi,$$
$$\textbf{(B)} \qquad \Box A_{AA'} + 2\Phi_{ABA'B'}A^{BB'} = 2q|\phi|^2 A_{AA'} + \mathrm{i}q\bar{\phi}\nabla_{AA'}\phi - \mathrm{i}q\phi\nabla_{AA'}\bar{\phi}.$$

These equations are supplemented by initial conditions for the values of ϕ and $A_{AA'}$ and of their normal derivatives.

Decomposition of the equations in the space-spinor formalism and the Newman-Penrose frame

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Space-spinor decomposition

Let $\tau^{AA'}$ denote the spinorial counterpart of a timelike vector field τ^a tangent to a congruence of curves. The Hermitian spinor $\tau^{AA'}$ is chosen to have the normalisation

$$\tau_{AA'}\tau^{AA'} = 2.$$

Consistent with the latter, we consider a spin dyad $\{o^A, \iota^A\}$ chosen so that

$$\tau^{AA'} = o^A\bar{o}^{A'} + \iota^A\bar{\iota}^{A'}.$$

It follows then that

$$\tau_{AA'}\tau^{BA'} = \delta_A{}^B.$$

The above relations induce a Hermitian conjugation operation via the relation

$$\mu^\dagger_A \equiv \tau_A{}^{A'}\bar{\mu}_{A'}.$$

The Newman-Penrose frame

The Newman-Penrose (NP) frame satisfies

$$\boldsymbol{g}(\boldsymbol{e}_{\boldsymbol{AA'}}, \boldsymbol{e}_{\boldsymbol{BB'}}) = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}},$$

of the form

$$\begin{aligned}
\boldsymbol{e}_{\boldsymbol{00'}} &= \frac{1}{\sqrt{2}}\Big((1-\tau)\boldsymbol{\partial}_\tau + \rho\boldsymbol{\partial}_\rho\Big),\\
\boldsymbol{e}_{\boldsymbol{11'}} &= \frac{1}{\sqrt{2}}\Big((1+\tau)\boldsymbol{\partial}_\tau - \rho\boldsymbol{\partial}_\rho\Big),\\
\boldsymbol{e}_{\boldsymbol{01'}} &= -\frac{1}{\sqrt{2}}\boldsymbol{X}_+,\\
\boldsymbol{e}_{\boldsymbol{10'}} &= -\frac{1}{\sqrt{2}}\boldsymbol{X}_-,
\end{aligned}$$

where $\boldsymbol{X}_+$ and $\boldsymbol{X}_-$ are vectors spanning the tangent space of $\mathbb{S}^2$ with dual covectors $\boldsymbol{\alpha}_+$ and $\boldsymbol{\alpha}_-$ such that metric of the 2-sphere is given by

$$\boldsymbol{\sigma} = 2(\boldsymbol{\alpha}^+ \otimes \boldsymbol{\alpha}^- + \boldsymbol{\alpha}^- \otimes \boldsymbol{\alpha}^+).$$

We define the *F-reduced wave operator* $\blacksquare$ acting on a scalar ζ as

$$\blacksquare\zeta \equiv (1-\tau^2)\ddot{\zeta} + 2\tau\rho\dot{\zeta}' - \rho^2\zeta'' - 2\tau\dot{\zeta} - \frac{1}{2}\eth\bar{\eth}\zeta - \frac{1}{2}\bar{\eth}\eth\zeta,$$

After some computations carried out using the suite `xAct` for tensorial and spinorial manipulations in the `Wolfram programming language`, the wave equations (**A**) and (**B**) are equivalent to the scalar system:

$$
\begin{aligned}
&(\mathbf{A}') && \blacksquare\phi = s,\\
&(\mathbf{B}') && \blacksquare\alpha_+ - 2\dot{\alpha}_+ = j_+,\\
&(\mathbf{C}') && \blacksquare\alpha_- + 2\dot{\alpha}_- = j_-,\\
&(\mathbf{D}') && \blacksquare\alpha_0 + \alpha_0 = j_0,\\
&(\mathbf{E}') && \blacksquare\alpha_2 + \alpha_2 = j_2.
\end{aligned}
$$

The scalars α, α_0, α_1 and α_2 have spin weight $0, -1, 0, 1$ whereas $\alpha_\pm$ and $j_\pm$ are defined by

$$\alpha_\pm \equiv \alpha \pm \alpha_1, \qquad j_\pm \equiv j \pm j_1,$$

all of them of spin-weight 0.

Key observation: the cylinder at infinity $\mathcal{I}$ is a *total characteristic* of the Maxwell-scalar field system.

This property can be used to construct, in a recursive manner, the *jets of order p at $\mathcal{I}$*, $J^p[\phi, \boldsymbol{\alpha}]$, $p \geq 0$ of the solutions to the evolution equations.

The jet is defined as

$$J^p[\phi, \boldsymbol{\alpha}] \equiv \{\partial_\rho^p \phi|_{\rho=0},\ \partial_\rho^p \boldsymbol{\alpha}|_{\rho=0}\}.$$

The F-reduced wave operator $\blacksquare$, reduces to an operator intrinsic to $\mathcal{I}$. Defining the *F-reduced wave operator on $\mathcal{I}$*, $\blacktriangle \equiv \blacksquare|_{\mathcal{I}}$, acting on a scalar ζ as

$$\blacktriangle\zeta \equiv (1-\tau^2)\ddot{\zeta} - 2\tau\dot{\zeta} - \frac{1}{2}(\eth\bar{\eth} + \bar{\eth}\eth)\zeta.$$

The decoupled case

Setting the charge parameter $\mathfrak{q} = 0$, the equations reduce to the linear system of equations

$$\begin{aligned}
&\blacksquare\phi = 0\\
&\blacksquare\alpha_+ = 0,\\
&\blacksquare\alpha_- = 0,\\
&\blacksquare\alpha_0 + 2\dot{\alpha}_0 = 0,\\
&\blacksquare\alpha_2 - 2\dot{\alpha}_2 = 0.
\end{aligned}$$

Defining

$$\phi^{(p)} \equiv \partial_\rho^p \phi|_{\rho=0}, \qquad \alpha_\pm^{(p)} \equiv \partial_\rho^p \alpha_\pm|_{\rho=0}, \qquad \alpha_0^{(p)} \equiv \partial_\rho^p \alpha_0|_{\rho=0}, \qquad \alpha_2^{(p)} \equiv \partial_\rho^p \alpha_2|_{\rho=0}, \qquad p \geq 0,$$

the order elements of $J^p[\phi, \boldsymbol{\alpha}]$ solutions satisfy a set of equations that can be summarised as three model equations:

$$\begin{aligned}
&(\mathbf{a}) \qquad \blacktriangle\zeta + 2p\tau\dot{\zeta} = 0,\\
&(\mathbf{b}) \qquad \blacktriangle\zeta + 2(p\tau + 1)\dot{\zeta} = 0,\\
&(\mathbf{c}) \qquad \blacktriangle\zeta + 2(p\tau - 1)\dot{\zeta} = 0.
\end{aligned}$$

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It is assumed that:
The scalar ζ in equation **(a)** has spin-weight 0 and admits an expansion of form

$$\zeta = \sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{m=-1}^{l}\frac{1}{p!}\zeta_{p;l,m}(Y_{lm})\rho^{p};$$

The scalar ζ in equation **(b)** has spin-weight 1 and admits an expansion of the form

$$\zeta = \sum_{p=1}^{\infty}\sum_{l=1}^{p}\sum_{m=-l}^{l}\frac{1}{p!}\zeta_{0p;l,m}({}_{1}Y_{lm})\rho^{p};$$

The scalar ζ in equation **(c)** has spin-weight -1 and an expansion of the form

$$\zeta = \sum_{p=1}^{\infty}\sum_{l=1}^{p}\sum_{m=-l}^{l}\frac{1}{p!}\zeta_{p;l,m}({}_{-1}Y_{lm})\rho^{p}.$$

Substituting the Ansatz into the model equations **(a)**-**(c)** one obtains the ordinary differential equations

$$\begin{aligned}
&\textbf{(a)} \quad (1-\tau^2)\ddot{\zeta}_{p;\ell,m} + 2\tau(p-1)\dot{\zeta}_{p;\ell,m} + (p+\ell)(\ell-p+1)\zeta_{p;\ell,m} = 0,\\
&\textbf{(b)} \quad (1-\tau^2)\ddot{\zeta}_{0,p;\ell,m} + 2\big((p-1)\tau+1\big)\dot{\zeta}_{0,p;\ell,m} + (p+\ell)(\ell-p+1)\zeta_{0,p;\ell,m} = 0,\\
&\textbf{(c)} \quad (1-\tau^2)\ddot{\zeta}_{2,p;\ell,m} + 2\big((p-1)\tau-1\big)\dot{\zeta}_{2,p;\ell,m} + (p+\ell)(\ell-p+1)\zeta_{2,p;\ell,m} = 0.
\end{aligned}$$

It can be verified that if $\zeta_{0,p;p,m}(\tau)$ is a solution to equation **(b)** then ${\zeta^{s}}_{0,p;p,m}(\tau) \equiv \zeta_{0,p;p,m}(-\tau)$ solves **(c)**. Thus, it is only necessary to study two model equations.

These equations are examples of *Jacobi ordinary differential equations*.

The behaviour of the solutions to equations **(a)**-**(c)** depends on the value of the parameter ℓ

For $0 \le \ell \le p-1$, $p \ge 1$ the nature of the solutions is summarised in the following:
lemma
The solutions to the system **(a)**-**(c)** can be written as

$$\zeta_{p;\ell,m}(\tau) = A_{p;\ell,m}\left(\frac{1-\tau}{2}\right)^{p} P_\ell^{(p,-p)}(\tau) + B_{p;\ell,m}\left(\frac{1+\tau}{2}\right)^{p} P_\ell^{(-p,p)}(\tau),$$

$$\zeta_{0,p;\ell,m}(\tau) = C_{p;\ell,m}\left(\frac{1-\tau}{2}\right)^{(p+1)} P_\ell^{(1+p,1-p)}(\tau) + D_{p;\ell,m}\left(\frac{1+\tau}{2}\right)^{(p-1)} P_\ell^{(-1-p,p-1)}(\tau),$$

$$\zeta_{2,p;\ell,m}(\tau) = E_{p;\ell,m}\left(\frac{1-\tau}{2}\right)^{(p+1)} P_\ell^{(1+p,1-p)}(-\tau) + F_{p;\ell,m}\left(\frac{1+\tau}{2}\right)^{(p-1)} P_\ell^{(-1-p,p-1)}(-\tau),$$

with $P_n^{(\alpha,\beta)}(\tau)$ Jacobi polynomials of order n and where

$$A_{p;\ell,m},\quad B_{p;\ell,m},\quad C_{p;\ell,m},\quad D_{p;\ell,m},\quad E_{p;\ell,m},\quad F_{p;\ell,m} \in \mathbb{C}$$

denote some constants which can be expressed in terms of the initial conditions.

However, for equation **(a)** in the case $\ell = p$ we have the following lemma:
lemma
For $l = p$ the solution to the equation **(a)** can be written as

$$\zeta_{p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{p}\left(\frac{1+\tau}{2}\right)^{p}\left(C_{1,p;\ell,m} + C_{2,p;\ell,m}\int_0^\tau \frac{\mathrm{d}s}{(1-s^2)^{p+1}}\right),$$

where $C_{1,p;\ell,m}$, $C_{2,p;\ell,m}$ are integration constants.

Letting $\zeta_{\star p;p,m} \equiv \zeta_{p;p,m}(0)$ and $\dot{\zeta}_{\star p;p,m} \equiv \dot{\zeta}_{p;p,m}(0)$ one finds that there is no logarithmic divergence if and only if $\dot{\zeta}_{\star p;p,m} = 0$ —that is, *when the initial data for ζ is time symmetric.*

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Similarly, equations **(b)** and **(c)** one obtains an analogous result:

lemma
For $\ell = p$ the solution to equations **(b)** and **(c)** can be written as

$$\zeta_{0,p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{(p+1)}\left(\frac{1+\tau}{2}\right)^{(p-1)}\left(C_{3,p;\ell,m} + C_{4,p;\ell,m}\int_0^\tau \frac{\mathrm{d}s}{(1-s)^{p+2}(1+s)^p}\right),$$
$$\zeta_{2,p;p,m}(\tau) = \left(\frac{1-\tau}{2}\right)^{(p-1)}\left(\frac{1+\tau}{2}\right)^{(p+1)}\left(C_{5,p;\ell,m} + C_{6,p;\ell,m}\int_0^\tau \frac{\mathrm{d}s}{(1+s)^{p+2}(1-s)^p}\right),$$

where $C_{3,p;\ell,m}$, $C_{4,p;\ell,m}$, $C_{5,p;\ell,m}$, $C_{6,p;\ell,m}$ are integration constants.

If we let $\zeta_{0,\star p;p,m} \equiv \zeta_{0,p;p,m}(0)$, $\dot{\zeta}_{0,\star p;p,m} \equiv \dot{\zeta}_{0,p;p,m}(0)$ and $\dot{\zeta}_{0\star,p;p,m} = -2\zeta_{0\star,p;p,m}$ there is no logarithmic divergence.

Now, since $a_s(\tau) \equiv a(-\tau)$ is a solution for the equation for $\zeta_{2,p;p,m}$ we have that to avoid logarithms in the solutions to equation **(c)** one needs the condition

$$\dot{\zeta}_{2\star,p;p,m} = 2\zeta_{2\star,p;p,m}.$$

Proposition

Given the jet $J^p[\phi, \boldsymbol{\alpha}]$ for $\mathfrak{q} = 0$ one has that:

(i) the elements of the jet have polynomial dependence in τ for the harmonic sectors with $0 \leq \ell \leq p-1$ and, thus, they extend analytically through $\tau = \pm 1$;

(ii) generically, for $\ell = p$, the solutions have logarithmic singularities at $\tau = \pm 1$. These logarithmic divergences can be precluded by fine-tuning of the initial data.

The coupled case

- The analysis of the case $\mathfrak{q} \neq 0$ begins by looking at the solutions corresponding to the jet $J^0[\phi, \boldsymbol{\alpha}]$ – that is $p = 0$.

The $p = 0$ order transport equations

Evaluating the system one finds that

$$
\begin{aligned}
&\textbf{(a')} \qquad \blacktriangle\phi^{(0)} = s^{(0)}, \\
&\textbf{(b')} \qquad \blacktriangle\alpha_+^{(0)} - 2\dot{\alpha}_+^{(0)} = j_+^{(0)}, \\
&\textbf{(c')} \qquad \blacktriangle\alpha_-^{(0)} + 2\dot{\alpha}_-^{(0)} = j_-^{(0)}, \\
&\textbf{(d')} \qquad \blacktriangle\alpha_0^{(0)} + \alpha_0^{(0)} = j_0^{(0)}, \\
&\textbf{(e')} \qquad \blacktriangle\alpha_2^{(0)} + \alpha_2^{(0)} = j_2^{(0)}.
\end{aligned}
$$

These transport equations decouple and it is possible to write down the solution explicitly. More precisely, one has that:

lemma

The unique solution to the 0-th order system **(a')-(e')** with suitable initial conditions is given by

$$\phi^{(0)} = \varphi_\star, \qquad \alpha_\pm^{(0)} = 0, \qquad \alpha_0^{(0)} = 0, \qquad \alpha_2^{(0)} = 0.$$

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The $p \geq 1$ transport equations

In order to analyse the properties of the jet of order p, $J^p[\phi, \boldsymbol{\alpha}]$ for given $p = n$, we assume that we have knowledge of the jets

$$J^0[\phi, \boldsymbol{\alpha}], \quad J^1[\phi, \boldsymbol{\alpha}], \ldots, J^{n-1}[\phi, \boldsymbol{\alpha}].$$

Under this assumption and taking into account the previous Lemma it follows that the elements of $J^p[\phi, \boldsymbol{\alpha}]$ satisfy the equations:

$$
\begin{aligned}
&\textbf{(a'')} \qquad \blacktriangle\phi^{(n)} + 2n\tau\dot{\phi}^{(n)} = s^{(n)},\\
&\textbf{(b'')} \qquad \blacktriangle\alpha_+^{(n)} + 2(n\tau - 1)\dot{\alpha}_+^{(n)} = 2q^2|\varphi_\star|^2\alpha_+^{(n)} + \tilde{j}_+^{(n)},\\
&\textbf{(c'')} \qquad \blacktriangle\alpha_-^{(n)} + 2(n\tau + 1)\dot{\alpha}_-^{(n)} = 2q^2|\varphi_\star|^2\alpha_-^{(n)} + \tilde{j}_-^{(n)},\\
&\textbf{(d'')} \qquad \blacktriangle\alpha_0^{(n)} + 2n\tau\dot{\alpha}_0^{(n)} + \alpha_0^{(n)} = 2q^2|\varphi_\star|^2\alpha_0^{(n)} + \tilde{j}_0^{(n)},\\
&\textbf{(e'')} \qquad \blacktriangle\alpha_2^{(n)} + 2n\tau\dot{\alpha}_2^{(n)} + \alpha_2^{(n)} = 2q^2|\varphi_\star|^2\alpha_2^{(n)} + \tilde{j}_2^{(n)},
\end{aligned}
$$

where $s^{(n)}$, $\tilde{j}_\pm^{(n)}$, $\tilde{j}_0^{(n)}$ and $\tilde{j}_2^{(n)}$ depend, solely, on the elements of $J^p[\phi, \boldsymbol{\alpha}]$, $0 \leq p \leq n-1$.

In particular, it is possible to consider model homogeneous equations of the form

$$\blacktriangle\zeta + 2(n\tau - 1)\dot{\zeta} - \varkappa\zeta = 0, \tag{1a}$$
$$\blacktriangle\zeta + 2n\tau\dot{\zeta} + (1 - \varkappa)\zeta = 0 \tag{1b}$$

with $\varkappa \equiv 2\mathfrak{q}^2|\varphi_\star|^2$. The solutions of these equations for generic choice of $\varkappa$ is radically different to that of the case $\varkappa = 0$ —i.e. $\mathfrak{q} = 0$.

Assuming that the various fields have an asymtptotic expansion as before, one is lead to consider a hierarchy of ordinary differential equations of the form

$$
\begin{aligned}
&\textbf{(a'')} && (1-\tau^2)\ddot{\phi}_{n;\ell,m} + 2(n-1)\tau\dot{\phi}_{n;\ell,m} + ((\ell-n+1)(n+\ell))\phi_{n;\ell,m} = s_{n;\ell,m},\\
&\textbf{(b'')} && (1-\tau^2)\ddot{\alpha}_{+,n;\ell,m} + 2(-1+(n-1)\tau)\dot{\alpha}_{+,n;\ell,m} + (\ell(\ell+1)-n(n-1)-\varkappa)\alpha_{+,n;\ell,m} = \tilde{j}_{+,n;\ell,m},\\
&\textbf{(c'')} && (1-\tau^2)\ddot{\alpha}_{-,n;\ell,m} + 2(1+(n-1)\tau)\dot{\alpha}_{-,n;\ell,m} + (\ell(\ell+1)-n(n-1)-\varkappa)\alpha_{-,n;l,m} = \tilde{j}_{-,n;\ell,m},\\
&\textbf{(d'')} && (1-\tau^2)\ddot{\alpha}_{0,n;\ell,m} + 2(n-1)\tau\dot{\alpha}_{0,n;\ell,m} + ((\ell-n+1)(n+\ell)-\varkappa)\alpha_{0,n;\ell,m} = \tilde{j}_{0,n;\ell,m},\\
&\textbf{(e'')} && (1-\tau^2)\ddot{\alpha}_{2,n;\ell,m} + 2(n-1)\tau\dot{\alpha}_{2,n;\ell,m} + ((\ell-n+1)(n+\ell)-\varkappa)\alpha_{2,n;\ell,m} = \tilde{j}_{2,n;\ell,m},
\end{aligned}
$$

for $0 \leq \ell \leq n$, $-\ell \leq m \leq \ell$ and with the *source terms*

$$
s_{n;\ell,m}, \quad \tilde{j}_{+,n;\ell,m}, \quad \tilde{j}_{-,n;\ell,m}, \quad \tilde{j}_{0,n;\ell,m}, \quad \tilde{j}_{2,n;\ell,m},
$$

known as a result of the spherical harmonics decomposition of the lower order jets $J^p[\phi, \boldsymbol{\alpha}]$ for $0 \leq p \leq n-1$.

For this, Frobenious's method is resorted to study the properties of the equations in terms of asymptotic expansions at the values $\tau = \pm 1$. The homogeneous version of equation **(a")**-**(e")** can be described in terms of the model equation

$$
(1-\tau^2)\ddot{\zeta} + 2(\varsigma + (n-1)\tau)\dot{\zeta} + (\ell(\ell+1) - n(n-1) - \varkappa)\zeta = 0
$$

where

$$
\varsigma = \begin{cases} -1 & \text{for} \quad \alpha_+ \\ 1 & \text{for} \quad \alpha_- \\ 0 & \text{for} \quad \phi,\ \alpha_0,\ \alpha_2 \end{cases}
$$

—recall also that $\varkappa = 2\mathfrak{q}^2\varphi_\star^2$.

Following Frobenius's method we look for power series solutions of the form

$$\zeta = (1-\tau)^r \sum_{k=0}^{\infty} D_k (1-\tau)^k, \qquad D_0 \neq 0. \tag{1}$$

Substiution of this Ansatz into the model equation leads to the *indicial equation*

$$2r(r-1) - 2\varsigma r - 2(n-1)r = 0.$$

ς	r_1	r_2
-1	0	$n-1$
0	0	n
1	0	$n+1$

Once the solutions to the indicial equation are known, the Ansatz leads to a recurrence relation for the coefficients D_k in the series.

The root $r_1 = 0$ of the indicial polynomial does not lead to a valid series solution.

We look for a second solution of the form

$$\zeta = \sum_{k=0}^{\infty} G_k (1-\tau)^k + (1-\tau)^{r_2} \log(1-\tau) \sum_{k=0}^{\infty} M_k (1-\tau)^k, \qquad G_0 \neq 0, \quad M_0 \neq 0.$$

The recurrence relations implied by the Ansatz shows that all the coefficients M_k for $k = 1, 2, \ldots$ and G_k for $k = 0, 1, 2, \ldots$ can be expressed in terms of the coefficient M_0.

Proposition

The general solution to

$$(1-\tau^2)\ddot{\zeta} + 2(\varsigma + (n-1)\tau)\dot{\zeta} + (\ell(\ell+1) - n(n-1) - \varkappa)\zeta = 0, \qquad \varkappa \neq 0$$

with $\varsigma = -1, 0, 1$ and $0 \leq \ell \leq n$, $n = 1, 2, \ldots$ consists, of:

(i) one solution which is analytic for $\tau \in [-1, 1]$;

(ii)one solution with is analytic for $\tau \in (-1, 1)$ and has logarithmic singularities at $\tau = \pm 1$. At these singular points the solution is of class $C^{r_2 - 1}$.

Main theorem

For generic data for Maxwell-scalar field system which have finite energy and are analytic around $\mathcal{I}$, the solution to the transport equations on $\mathcal{I}$ develop logarithmic singularities at the critical points $\mathcal{I}^+$ and $\mathcal{I}^-$.

The nonlinear interaction makes both fields more singular than what is seen when the fields are non-interacting!

Thank you for your attention