Nonlocal Charges in Linearized Gravity

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Construction

Linearized GR

- *Einstein-Hilbert action*

$$\mathcal{S}_{EH} = \frac{1}{2\kappa}\int_{\mathcal{M}} R\,\boldsymbol{\epsilon}\ , \qquad \kappa = \frac{8\pi G}{c^4}$$

- *linearized metric*

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\ , \qquad |h_{\mu\nu}| \ll 1$$

- *linearized Lagrangian*

$$\mathcal{L}_{\rm LG} = \tfrac{1}{2}\left(\partial_\beta h^\alpha{}_\alpha\,\partial^\beta h^\gamma{}_\gamma - 2\,\partial_\beta h^\alpha{}_\alpha\,\partial^\gamma h^\beta{}_\gamma - \partial_\gamma h_{\alpha\beta}\,\partial^\gamma h^{\alpha\beta} + 2\,\partial_\gamma h_{\alpha\beta}\,\partial^\beta h^{\alpha\gamma}\right)$$

- *in transverse-traceless gauge*

$$h_{0\alpha} = 0\ , \quad h_{ij}{}^{,j} = 0\ , \quad h^i{}_i = 0$$

- *by adding a total derivative term*

$$\mathcal{L}'_{\rm LG} = \mathcal{L}_{\rm LG} - \tfrac{1}{2}\left(h_{jk}\,h^{ik,j}\right)_{,i} \; = \tfrac{1}{2}\left(\dot{h}_{ij}\,\dot{h}^{ij} - h_{ij,k}\,h^{ij,k} + \, h_{ij,k}\,h^{ik,j}\right)$$

$$= \tfrac{1}{2}\left[\dot{\boldsymbol{h}}\cdot\dot{\boldsymbol{h}} - (\nabla\times\boldsymbol{h})\cdot(\nabla\times\boldsymbol{h})\right]$$

Symmetric 2-Tensor Operations

- *the scalar dot product* $\boldsymbol{c} \cdot \boldsymbol{d} = c_{ij}\, d^{ij}$,
- *the cross product* $(\boldsymbol{c} \times \boldsymbol{d})_i = \epsilon_i{}^{jk}\, c_{jl}\, d_k{}^l$,
- *2-tensor dot product* $(\boldsymbol{c} : \boldsymbol{d})_{ij} = c_{k(i}\, d_{j)}{}^k$,
- *the wedge product* $(\boldsymbol{c} \wedge \boldsymbol{d})_{ij} = \epsilon_i{}^{kl}\, \epsilon_j{}^{mn}\, c_{km}\, d_{ln}$
- *divergence of a 2-tensor* $(\nabla \cdot \boldsymbol{e})_i = e_{ij}{}^{,j}$
- *curl of a 2-tensor* $(\nabla \times \boldsymbol{e})_{ij} = \epsilon_{(i}{}^{kl}\, e_{j)l,k}$

Analogy to Maxwell Theory

- *analogues of the electric and magnetic fields*

$$e_{ij} = -\dot{h}_{ij} \ , \quad b_{ij} = \epsilon_i{}^{lm} \, h_{jm,l}$$

- *similar Lagrangian and action*

$$\begin{aligned} \mathcal{L}'_{\text{LG}} &= \frac{1}{2}\left[\dot{\boldsymbol{h}} \cdot \dot{\boldsymbol{h}} - (\nabla \times \boldsymbol{h}) \cdot (\nabla \times \boldsymbol{h})\right] \\ &= \frac{1}{2}\left(\boldsymbol{e} \cdot \boldsymbol{e} - \boldsymbol{b} \cdot \boldsymbol{b}\right) , \end{aligned}$$

- *similar equations of motion*

$$\nabla \cdot \boldsymbol{e} = 0 \ , \qquad \nabla \cdot \boldsymbol{b} = 0$$

$$\nabla \times \boldsymbol{e} = -\dot{\boldsymbol{b}} \ , \qquad \nabla \times \boldsymbol{b} = \dot{\boldsymbol{e}}$$

Relation to Wyle Curvature

- *decomposition of the Weyl tensor into 'electric' and 'magnetic' parts:*

$$E_{ij} = C_{i0j0} \quad , \quad B_{ij} = {}^{*}C_{i0j0}$$

- *from Gauss-Codazzi equations for vacuum spacetime*

$$-(\nabla \times K)_{ij} = B_{ij}$$
$$^{3}R_{ij} - K_{im}\,K^{m}{}_{j} + K_{ij}\,tr K = E_{ij}$$

  – *second fundamental form (extrinsic curvature) of time slices*

$$K_{ij} = -\tfrac{1}{2}\,\dot{h}_{ij} = -\tfrac{1}{2}\,e_{ij}$$

  – *Ricci tensor of time slices*

$$^{3}R_{ij} = -\tfrac{1}{2}\,\partial_k \partial^k\,h_{ij} = \tfrac{1}{2}\,(\nabla \times \boldsymbol{b})_{ij}$$

Relation to Wyle Curvature

- *using the linearized field equations, we find*

$$\frac{1}{2}\nabla\times\boldsymbol{e}=\boldsymbol{B} \quad , \quad \frac{1}{2}\nabla\times\boldsymbol{b}=\boldsymbol{E}$$

> *nonlocal definition of the fields e, b*

$$\boldsymbol{e}=2\,\nabla^{-2}\,\nabla\times\boldsymbol{B} \quad , \quad \boldsymbol{b}=2\,\nabla^{-2}\,\nabla\times\boldsymbol{E}$$

– *where*

$$\nabla^{-2}\,\nabla\times\boldsymbol{B}=\int\frac{\nabla\times\boldsymbol{B}}{|\boldsymbol{x}-\boldsymbol{x}'|}\frac{d^3x'}{4\pi}$$

is a generalization of the Biot-Savart operator to 2-tensors

- *interpreting* h *as nonlocally constructed from* b *:* ***gauge invariant***

1st nonlocality aspect

Duality-symmetric LG

- *introducing a second (auxiliary) gravitational potential k*
- *duality-symmetric Lagrangian*

$$\mathcal{L}_{\text{LG-ds}} = \frac{1}{4}\left[\dot{\boldsymbol{h}} \cdot \dot{\boldsymbol{h}} - (\nabla \times \boldsymbol{h}) \cdot (\nabla \times \boldsymbol{h}) + \dot{\boldsymbol{k}} \cdot \dot{\boldsymbol{k}} - (\nabla \times \boldsymbol{k}) \cdot (\nabla \times \boldsymbol{k})\right]$$

- *E&M-analogous fields in terms of potentials*

$$\boldsymbol{e} = -\nabla \times \boldsymbol{k}$$
$$\boldsymbol{b} = \nabla \times \boldsymbol{h}\,,$$

- *duality constraint*

$$\dot{\boldsymbol{h}} = \nabla \times \boldsymbol{k}$$
$$\dot{\boldsymbol{k}} = -\nabla \times \boldsymbol{h}$$

2nd nonlocality aspect

Charges

Helicity

- *helicity is the associated conserved quantity with the duality symmetry*

$$h_{ij} \to h_{ij} \, \cos\theta + k_{ij} \, \sin\theta \ , \quad k_{ij} \to k_{ij} \, \cos\theta - h_{ij} \, \sin\theta$$

  - *helicity (Noether) current:* $$J^\alpha = \frac{\partial \mathcal{L}_{\text{LG-ds}}}{\partial (h_{ij,\alpha})} \, k_{ij} - \frac{\partial \mathcal{L}_{\text{LG-ds}}}{\partial (k_{ij,\alpha})} \, h_{ij} \ , \quad \partial_\alpha \, J^\alpha = 0$$

  - *helicity density:* $$\mathcal{H} \equiv 2 \, J^0 = \boldsymbol{h} \cdot \boldsymbol{b} - \boldsymbol{k} \cdot \boldsymbol{e} \ ,$$

  - *helicity flux:* $$\boldsymbol{S} \equiv 2 \, \boldsymbol{J} = \boldsymbol{e} \times \boldsymbol{h} + \boldsymbol{b} \times \boldsymbol{k}$$

Helicity and Biot-Savart Operator

- *Writhe of a smooth curve*

$$Wr(\gamma) = \int_{\gamma\times\gamma} \left(\frac{d\boldsymbol{x}}{ds} \times \frac{d\boldsymbol{x}'}{ds'}\right) \cdot \frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\, \mathrm{d}s\, \mathrm{d}s'$$

  - *describes the amount of "coiling" of a knot or any closed curve in 3-space*
  - *the standard measure of the extent that the curve wraps and coils around itself*

- *helicity H(v) of a smooth vector field v on a domain Ω in 3-space*

$$H(\boldsymbol{v}) = \frac{1}{4\pi}\int_{\Omega\times\Omega} \boldsymbol{v}(\boldsymbol{x}) \times \boldsymbol{v}(\boldsymbol{x}') \cdot \frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\, \mathrm{d}^3x\, \mathrm{d}^3x'$$

$$= \int_{\Omega} \boldsymbol{v} \cdot BS(\boldsymbol{v})\, \mathrm{d}^3x$$

Energy-mometum

- *canonical energy-momentum tensor*

$$T_\alpha{}^\beta = \delta_\alpha{}^\beta\, \mathcal{L}_{\text{LG-ds}} - \frac{\partial \mathcal{L}_{\text{LG-ds}}}{\partial(h_{ij,\beta})}\, h_{ij,\alpha} - \frac{\partial \mathcal{L}_{\text{LG-ds}}}{\partial(k_{ij,\beta})}\, k_{ij,\alpha}\ , \quad \partial_\beta\, T_\alpha{}^\beta = 0$$

- *energy, energy flux, momentum and spatial stress*

$$-T_0{}^0 \equiv \mathcal{E} = \tfrac{1}{2}\,(\dot h_{ij}\dot h^{ij} + \dot k_{ij}\dot k^{ij}) = \tfrac{1}{2}\,(\boldsymbol{e}\cdot\boldsymbol{e} + \boldsymbol{b}\cdot\boldsymbol{b})$$

$$-T_0{}^i \equiv P^i = -\tfrac{1}{2}\,(\epsilon^{ijk}\, b_{nj}\, \dot h^n{}_k - \epsilon^{ijk}\, e_{nj}\dot k^n{}_k) = (\boldsymbol{e}\times\boldsymbol{b})_i$$

$$T_i{}^0 \equiv P_i^o = -\tfrac{1}{2}\,(\dot h^{jk}\, h_{jk,i} + \dot k^{jk}\, k_{jk,i}) = \tfrac{1}{2}\,[\boldsymbol{e}\cdot(\nabla)\boldsymbol{h} + \boldsymbol{b}\cdot(\nabla)\boldsymbol{k}]_i$$

$$T_i{}^j \equiv \sigma_i{}^j = \tfrac{1}{2}\,(-\epsilon^{jkl}\, b^n{}_k\, h_{nl,i} + \epsilon^{jkl}\, e^n{}_k\, k_{nl,i})$$

where

$$[\boldsymbol{e}\cdot(\nabla)\boldsymbol{h}]_i = e^{jk}\, h_{jk,i}$$

Spin and Orbital Angular Momentum

- *spin & orbital angular momenta*

$$M^{\alpha\beta\gamma} = \tilde{L}^{\alpha\beta\gamma} + \tilde{S}^{\alpha\beta\gamma} \;, \quad \partial_\alpha \, M^{\alpha\beta\gamma} = 0$$

- *orbital part*

$$\tilde{L}^{\alpha\beta\gamma} = r^\alpha T^{\beta\gamma} - r^\beta T^{\alpha\gamma}$$

$$\tilde{L}^{ij0} = e_{kl} \, h^{kl,[j} \, r^{i]} + b_{kl} \, k^{kl,[j} \, r^{i]}$$

$$\tilde{L}^{ijk} = \epsilon_{lmn} \, b_p^{\;m} \, h^{pn,[i} \, r^{j]} + \epsilon_{lmn} \, e_p^{\;m} \, k^{pn,[i} \, r^{j]}$$

- *spin part*

$$\tilde{S}_{ij}{}^{\gamma} = \frac{\partial \mathcal{L}_{\text{LG-ds}}}{\partial(\partial_\gamma h_{mn})} \left[(\mathcal{M}_{ij})_{ml} \, h_{ln} + (\mathcal{M}_{ij})_{nl} \, h_{ml} \right] + \frac{\partial \mathcal{L}_{\text{LG-ds}}}{\partial(\partial_\gamma k_{mn})} \left[(\mathcal{M}_{ij})_{ml} \, k_{ln} + (\mathcal{M}_{ij})_{nl} \, k_{ml} \right]$$

$$\tilde{S}_{ij}{}^{0} = \frac{1}{2} \left(e_{n[i} \, h_{j]n} + b_{n[i} \, k_{j]n} \right)$$

$$\tilde{S}_{ij}{}^{k} = \frac{1}{4} \left(\epsilon_{kln} \, h_{l[i} \, b_{j]n} - \epsilon_{kl[i} \, h_{j]n} \, b_{nl} - \epsilon_{kln} \, k_{l[i} \, e_{j]n} + \epsilon_{kl[i} \, k_{j]n} \, e_{nl} \right)$$

Summary and Directions

- *using a Maxwell-like duality-symmetric Lagrangian for linearized gravity*
- *canonical charges: helicity, energy-momentum and angular momentum*
- *2 aspects of nonlocality*

? generalization to other backgrounds

? characterization of gravitational waves


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