Necessary Conditions for the Existence of the Carter Constant in a General Axially Symmetric Stationary Spacetime in Plasma and Applications

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It is useful to apply the Hamiltonian formalism with Hamiltonian in the form[1]

$$\mathcal{H}(x^\alpha, p_\alpha) = \frac{1}{2}\left[g^{\beta\delta}p_\beta p_\delta + \omega_{pl}^2(x^\alpha)\right],$$

where $g^{\beta\delta}$ is the spacetime metric, x^α are the spacetime coordinates, p_α denotes the wave vector, and

$$\omega_{pl}^2 = \frac{e^2 N(x)}{\epsilon_0 m_e}$$

is the plasma electron frequency. To propagate through a medium, wave frequency $\omega(x^\alpha)$ must obey $\omega(x^\alpha) > \omega_{pl}(x^\alpha)$.

Hamilton equations of motion are

$$\frac{dx^\alpha}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_\alpha}, \quad \frac{dp_\alpha}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^\alpha}.$$

[1] Based on Synge [1960].

An arbitrary axially symmetric stationary object can be described by the metric in a general form

$$\begin{aligned} ds^2 = &- A(x_1,x_2)dk_1^2 + B(x_1,x_2)dx_1^2 + 2P(x_1,x_2)dk_1dk_2 + \\ &+ C(x_1,x_2)dk_2^2 + D(x_1,x_2)dx_2^2. \end{aligned}$$

$B(x_1,x_2), D(x_1,x_2)$ – positive functions of x_1 and x_2,
$A(x_1,x_2), C(x_1,x_2)$, and $P(x_1,x_2)$ – generally of an arbitrary sign
Notice, e.g. dependence on angular momentum a.

Let us further assume that plasma frequency $\omega_{pl}^2(x^\alpha)$ is solely function of coordinates x_1 and x_2, i.e. $\omega_{pl}^2(x_1,x_2)$.

Constants of Motion

From the equations of motion one gets

$$\frac{\partial \mathcal{H}}{\partial k_1} = 0 \quad \rightarrow \quad p_{k_1},$$
$$\frac{\partial \mathcal{H}}{\partial k_2} = 0 \quad \rightarrow \quad p_{k_2}.$$

Let us further denote $p_{k_1} \equiv -\omega_0$, $p_{k_2} \equiv p_\kappa$.
It holds

$$\mathcal{H}(x^\alpha, p_\alpha) = 0.$$

To get a closed (separated) system of equations, one more parameter is needed – Carter constant $\mathcal{K}$.

$\mathcal{H}(x^\alpha, p_\alpha) \to \mathcal{H}(x^\alpha, \frac{\partial S}{\partial x^\alpha})$
Assume S can be separated as follows:

$$S(k_1, k_2, x_1, x_2) = -\omega_0 k_1 + p_\kappa k_2 + S_{x_1}(x_1) + S_{x_2}(x_2).$$

Then, the Hamilton-Jacobi equation takes the form

$$0 = \frac{1}{2}\left[\frac{1}{B(x_1,x_2)}\left(\frac{\partial S_{x_1}}{\partial x_1}\right)^2 + \frac{1}{D(x_1,x_2)}\left(\frac{\partial S_{x_2}}{\partial x_2}\right)^2 + \omega_{pl}^2(x_1,x_2) + \right.$$
$$\left. + \frac{p_\kappa^2 A(x_1,x_2) - \omega_0^2 C(x_1,x_2) - 2\omega_0 p_\kappa P(x_1,x_2)}{A(x_1,x_2)C(x_1,x_2) + P^2(x_1,x_2)}\right]. \qquad (1)$$

To obtain terms associated with partial derivatives of S_{x_1}, S_{x_2} solely as functions of x_1 and x_2, let us multiply (1) by an arbitrary function $F(x_1, x_2)$ to get

$$0 = \frac{1}{2}\left[\frac{F(x_1,x_2)}{B(x_1,x_2)}\left(\frac{\partial S_{x_1}}{\partial x_1}\right)^2 + \frac{F(x_1,x_2)}{D(x_1,x_2)}\left(\frac{\partial S_{x_2}}{\partial x_2}\right)^2 + \right.$$

$$+ \frac{F(x_1,x_2)\left(p_\kappa^2 A(x_1,x_2) - \omega_0^2 C(x_1,x_2) - 2\omega_0 p_\kappa P(x_1,x_2)\right)}{A(x_1,x_2)C(x_1,x_2) + P^2(x_1,x_2)} +$$

$$\left. + F(x_1,x_2)\omega_{pl}^2(x_1,x_2)\right]$$

Function $F(x_1, x_2)$ has to be such that

$$\frac{F(x_1, x_2)}{B(x_1, x_2)} \equiv \mathcal{F}(x_1) \quad \wedge \quad \frac{F(x_1, x_2)}{D(x_1, x_2)} \equiv \mathcal{G}(x_2)$$
$$\rightarrow \quad \frac{B(x_1, x_2)}{D(x_1, x_2)} = \frac{\mathcal{G}(x_2)}{\mathcal{F}(x_1)}.$$

Moreover,

$$\frac{F(x_1, x_2)}{A(x_1, x_2)C(x_1, x_2) + P^2(x_1, x_2)} X(x_1, x_2) = X_{x_1} + X_{x_2},$$

where $X(x_1, x_2)$ stands for $A(x_1, x_2)$, $C(x_1, x_2)$, and $P(x_1, x_2)$, and

$$\omega_{pl}^2(x_1, x_2) \equiv \frac{f_{x_1}(x_1) + f_{x_2}(x_2)}{F(x_1, x_2)}.$$

Using the new forms of corresponding terms gives

$$\mathcal{F}(x_1)\left(\frac{\partial S_{x_1}}{\partial x_1}\right)^2 + f_{x_1}(x_1) + p_\kappa^2 A_{x_1} - \omega_0^2 C_{x_1} - 2\omega_0 p_\kappa P_{x_1}$$

$$= -\mathcal{G}(x_2)\left(\frac{\partial S_{x_2}}{\partial x_2}\right)^2 - f_{x_2}(x_2) - p_\kappa^2 A_{x_2} + \omega_0^2 C_{x_2} + 2\omega_0 p_\kappa P_{x_2} \equiv -\mathcal{K}.$$

Starting from the Hamilton equation

$$\dot{x}_1 = \frac{\partial \mathcal{H}}{\partial p_{x_1}} = \frac{p_{x_1}}{B(x_1, x_2)},$$

we find

$$B^2(x_1, x_2)\mathcal{F}(x_1)\dot{x}_1^2 = -\mathcal{K} - f_{x_1} - p_\kappa^2 A_{x_1} + \omega_0^2 C_{x_1} + 2\omega_0 p_\kappa P_{x_1} \equiv R(x_1)$$

Condition $\dot{x}_1 = \ddot{x}_1 = 0$ leads to

$$p_\kappa = \frac{\omega_0 P'_{x_1}}{A'_{x_1}} \left(1 \pm \sqrt{1 - \frac{A'_{x_1}}{P'^2_{x_1}} \left(\frac{f'_{x_1}}{\omega_0^2} - C'_{x_1}\right)}\right),$$

$$\mathcal{K} = \frac{A_{x_1}}{A'_{x_1}} \left(f'_{x_1} - \omega_0^2 C'_{x_1}\right) + \omega_0^2 C_{x_1} + 2\frac{\omega_0^2 P'_{x_1}}{A'_{x_1}} \left(P_{x_1} - \frac{A_{x_1} P'_{x_1}}{A'_{x_1}}\right) \times \left(1 \pm \sqrt{1 - \frac{A'_{x_1}}{P'^2_{x_1}} \left(\frac{f'_{x_1}}{\omega_0^2} - C'_{x_1}\right)}\right) - f_{x_1}.$$

From

$$\dot{x}_2 = \frac{\partial \mathcal{H}}{\partial p_{x_2}} = \frac{p_{x_2}}{D(x_1, x_2)}$$

$$\rightarrow \quad D^2(x_1, x_2)\mathcal{G}(x_2)\dot{x}_2^2 = \mathcal{K} - f_{x_2} - p_\kappa^2 A_{x_2} + \omega_0^2 C_{x_2} + 2\omega_0 p_\kappa P_{x_2},$$

so it has to hold

$$\mathcal{K} - f_{x_2} \geq p_\kappa^2 A_{x_2} - \omega_0^2 C_{x_2} - 2\omega_0 p_\kappa P_{x_2}. \quad (2)$$

Notice that we can omit the multiplication of function $\mathcal{G}(x_2)$ on both sides of formula (2).

Plugging the expressions for $\mathcal{K}$ and p_κ obtained above leads to

$$\frac{A_{x_1}}{A'_{x_1}}\left(\frac{f'_{x_1}}{\omega_0^2}-C'_{x_1}\right)+C_{x_1}-\frac{f_{x_1}+f_{x_2}}{\omega_0^2}+$$
$$+2\frac{P'_{x_1}}{A'_{x_1}}\left(P_{x_1}-\frac{A_{x_1}P'_{x_1}}{A'_{x_1}}\right)\left(1\pm\sqrt{1-\frac{A'_{x_1}}{P'^2_{x_1}}\left(\frac{f'_{x_1}}{\omega_0^2}-C'_{x_1}\right)}\right)\geq$$
$$-\frac{A_{x_2}}{A'_{x_1}}\left(\frac{f'_{x_1}}{\omega_0^2}-C'_{x_1}\right)-C_{x_2}+$$
$$+2\frac{P'_{x_1}}{A'_{x_1}}\left(\frac{A_{x_2}P'_{x_1}}{A'_{x_1}}-P_{x_2}\right)\left(1\pm\sqrt{1-\frac{A'_{x_1}}{P'^2_{x_1}}\left(\frac{f'_{x_1}}{\omega_0^2}-C'_{x_1}\right)}\right).$$

At every point (x_1, x_2) where this condition holds, there exists a spherical light ray and the photon region is formed.

The Orthonormal Tetrad

Let us introduce an observer located at (x_{1O}, x_{2O}) with well-defined radial coordinate (i.e. outside the black hole horizon) and an orthonormal tetrad

$$
\begin{aligned}
e_0 &= \left.(Y_1 \partial_{k_1} + Y_2 \partial_{k_2})\right|_{(x_{1O}, x_{2O})}, \\
e_1 &= \left.\frac{1}{\sqrt{D(x_1, x_2)}} \partial_{x_2}\right|_{(x_{1O}, x_{2O})}, \\
e_2 &= \left.(Y_3 \partial_{k_1} + Y_4 \partial_{k_2})\right|_{(x_{1O}, x_{2O})}, \\
e_3 &= -\left.\frac{1}{\sqrt{B(x_1, x_2)}} \partial_{x_1}\right|_{(x_{1O}, x_{2O})}.
\end{aligned}
$$

Coefficients Y_1, Y_2, Y_3, Y_4 are chosen such that the orthonormality conditions $g(e_0, e_0) = -1$, $g(e_2, e_2) = 1$, $g(e_0, e_2) = 0$ hold.

The tangent vector of light ray $\lambda(s)$ is

$$\dot{\lambda} = \dot{x}_1 \partial_{x_1} + \dot{x}_2 \partial_{x_2} + \dot{k}_1 \partial_{k_1} + \dot{k}_2 \partial_{k_2}.$$

Moreover, there is also the tangent vector at the observation event

$$\dot{\lambda} = -\alpha e_0 + \beta(\sin\theta \cos\psi e_1 + \sin\theta \sin\psi e_2 + \cos\theta e_3).$$

α, β – positive factors
θ, ψ – the colatitude and the azimuthal angle (the celestial coordinates of the observer)

The light rays are parameterized as $g(\dot{\lambda}, \dot{\lambda}) = -\omega_{pl}^2$ and thus

$$\alpha^2 - \beta^2 = \omega_{pl}^2\big|_{(x_{1O}, x_{2O})} .$$

Furthermore, α can be derived as

$$\begin{aligned} \alpha = g(\dot{\lambda}, e_0) &= Y_1(\dot{k}_1 g_{k_1 k_1} + \dot{k}_2 g_{k_1 k_2}) + Y_2(\dot{k}_1 g_{k_1 k_2} + \dot{k}_2 g_{k_2 k_2}) \\ &= Y_1(-\omega_0) + Y_2 p_\kappa, \end{aligned}$$

and then

$$\beta = \sqrt{(-Y_1 \omega_0 + Y_2 p_\kappa)^2 - \omega_{pl}^2}.$$

Comparing factors of ∂_{x_1} and ∂_{k_2} in different expressions of $\dot{\lambda}$ yields

$$\dot{x}_1 = -\beta \cos\theta \frac{1}{\sqrt{B(x_1, x_2)}},$$
$$\dot{k}_2 = -\alpha Y_2 + \beta \sin\theta \sin\psi Y_4.$$

After plugging into these general formulae, one gets

$$\sin\theta = \left(1 + \frac{\mathcal{K} + f_{x_1} + p_\kappa^2 A_{x_1} - \omega_0^2 C_{x_1} - 2\omega_0 p_\kappa P_{x_1}}{F(x_1, x_2)((-Y_1\omega_0 + Y_2 p_\kappa)^2 - \omega_{pl}^2)}\right)^{1/2}\Bigg|_{(x_{1O}, x_{2O})},$$

$$\sin\psi = \frac{(A_{x_1} + A_{x_2} + F(x_1, x_2)Y_2^2)p_\kappa - (P_{x_1} + P_{x_2} + F(x_1, x_2)Y_1 Y_2)\omega_0}{F(x_1, x_2)Y_4\left[F(x_1, x_2)\left(-Y_1\omega_0 + Y_2 p_\kappa\right)^2 + \mathcal{K} - f_{x_2} + p_\kappa^2 A_{x_1} - \omega_0^2 C_{x_1} - 2\omega_0 p_\kappa P_{x_1}\right]^{1/2}}$$

Conclusions

- General conditions for the existence of the Carter constant in an axially symmetric stationary spacetime in plasma were derived.
- The necessary conditions on the form of the spacetime metric and plasma density are independent.
- General forms of the photon region and black hole shadow under defined conditions were obtained.

- Kerr Metric (Perlick and Tsupko, 2017)
- Hartle-Thorne Metric

In both cases
$x_1 = r$, $x_2 = \vartheta$, $k_1 = t$, $k_2 = \varphi$.

$$B(r,\vartheta) = \frac{\rho^2}{\Delta}, \quad D(r,\vartheta) = \rho^2,$$
$$F(r,\vartheta) = \rho^2, \quad \mathcal{F}(r) = \Delta, \quad \mathcal{G}(\vartheta) = 1,$$

$$A_r = -\frac{a^2}{\Delta}, \qquad C_r = \frac{(r^2+a^2)^2}{\Delta}, \quad P_r = -\frac{a(r^2+a^2)}{\Delta},$$
$$A_\vartheta = \sin^{-2}\vartheta, \qquad C_\vartheta = -a^2\sin^2\vartheta, \quad P_\vartheta = a,$$

where $\Delta = r^2 + a^2 - 2Mr$, $\rho^2 = r^2 + a^2\cos^2\vartheta$.

Assuming that $\omega_{pl}^2(r,\vartheta) = (f_r + f_\vartheta)/F(r,\vartheta)$, applying the general formulae gives

$$\mathcal{F}(r)\left(\frac{\partial S_r}{\partial r}\right)^2 + f_r + p_\varphi^2 A_r - \omega_0^2 C_r - 2\omega_0 p_\varphi P_r =$$

$$\Delta\left(\frac{\partial S_r}{\partial r}\right)^2 + f_r - \frac{1}{\Delta}(a p_\varphi + (r^2+a^2)\omega_0)^2,$$

and

$$-\mathcal{G}(\vartheta)\left(\frac{\partial S_\vartheta}{\partial \vartheta}\right)^2 - f_\vartheta - p_\varphi^2 A_\vartheta + \omega_0^2 C_\vartheta + 2\omega_0 p_\varphi P_\vartheta =$$

$$-\left(\frac{\partial S_\vartheta}{\partial \vartheta}\right)^2 - f_\vartheta - \left(\frac{p_\varphi}{\sin\vartheta} + a\sin\vartheta\omega_0\right)^2.$$

See (27) in Perlick and Tsupko (2017).

$$A(r,\vartheta) = A_1 \mathcal{J}_A - j_1^2 \sin^2\vartheta r^2 \mathcal{J}_\varphi, \qquad B(r,\vartheta) = A_1^{-1} \mathcal{J}_B,$$
$$C(r,\vartheta) = \sin^2\vartheta r^2 \mathcal{J}_\varphi, \qquad D(r,\vartheta) = r^2 \mathcal{J}_\varphi,$$
$$P(r,\vartheta) = -j_1 \sin^2\vartheta r^2 \mathcal{J}_\varphi, \qquad A_1 = 1 - \frac{2M}{r} + \frac{2J^2}{r^4},$$

where

$$\mathcal{J}_A = \left\{1 + 2P_2(\cos\vartheta)\left[j\left(1+\frac{M}{r}\right) + KQ_2^2\right]\right\},$$
$$\mathcal{J}_B = \left\{1 - 2P_2(\cos\vartheta)\left[j\left(1-\frac{5M}{r}\right) + KQ_2^2\right]\right\},$$
$$\mathcal{J}_\varphi = \left\{1 + 2P_2(\cos\vartheta)\left[-j\left(1+\frac{2M}{r}\right) + K\left(\frac{2M}{\sqrt{r(r-2M)}}Q_2^1 - Q_2^2\right)\right]\right\}.$$

It can be seen that the ratio of terms $B(r,\vartheta)$ and $D(r,\vartheta)$ gives

$$\frac{B(r,\vartheta)}{D(r,\vartheta)} = \frac{\mathcal{J}_B}{A_1 r^2 \mathcal{J}_\varphi} \neq \frac{\mathcal{G}(\vartheta)}{\mathcal{F}(r)}.$$