Equipotential Photon Surface Uniqueness in Electrovacuum

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joint work with

S. Borghini and *C. Cederbaum*

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We study solutions $(\mathcal{L}^4, g)$ to the **source–free Einstein–Maxwell equations** that are

1. **Standard Static**
   $\exists\ (\mathrm{M}, g_0)$ Riemannian manifold with compact boundary $\partial\mathrm{M}$ and $N : \mathrm{M} \to \mathbb{R}$ with $N > 0$ in $\mathring{\mathrm{M}}$, called *Lapse Function* such that

$$\mathcal{L}^4 = \mathbb{R} \times \mathrm{M}, \qquad g = -N^2 dt^2 + g_0.$$

2. **Asymptotically Flat**

in presence of (a Black Hole Horizon or) an Equipotential Photon Surface.

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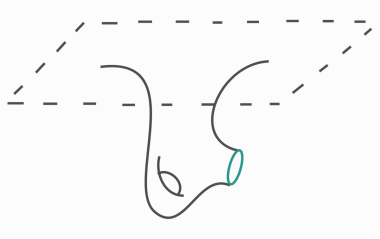

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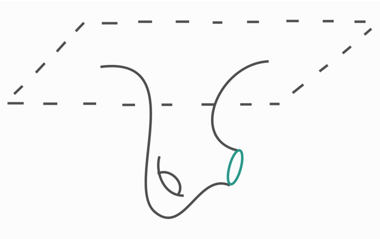

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Introducing the *Electric Potential* $\Psi : \mathrm{M} \to \mathbb{R}$, the problem can be reduced to the study of tuples $(\mathrm{M}, g_0, N, \Psi)$ which satisfy

$$\begin{cases} N \,\mathrm{Ric} \,=\, \mathrm{D}^2 N \,-\, \dfrac{2}{N}\, \mathrm{d}\Psi^2 + \dfrac{1}{N}\, |\mathrm{D}\Psi|^2\, g_0 & \text{in } \mathrm{M}\,, \\ \Delta N \,=\, \dfrac{1}{N}\, |\mathrm{D}\Psi|^2 & \text{in } \mathrm{M}\,, \\ \Delta \Psi \,=\, \dfrac{1}{N}\, g_0(\mathrm{D}N\,,\, \mathrm{D}\Psi) & \text{in } \mathrm{M}\,, \\ & \\ N = N_0 \geq 0, \quad \Psi = \Psi_0 > 0 & \text{on } \partial\mathrm{M} \end{cases} \tag{1}$$

and the decay conditions

$$N \,=\, 1 - \frac{m_{ADM}}{|\,x\,|} + o_2(|\,x\,|^{-1}), \qquad \Psi = o(1) \qquad \text{as } |\,x\,| \to +\infty\,,$$

The Reissner-Nordström solution

The unique **rotationally symmetric** solution is the *Reissner-Nordström solution* of *mass* $m \in \mathbb{R}$ and *charge* $q > 0$

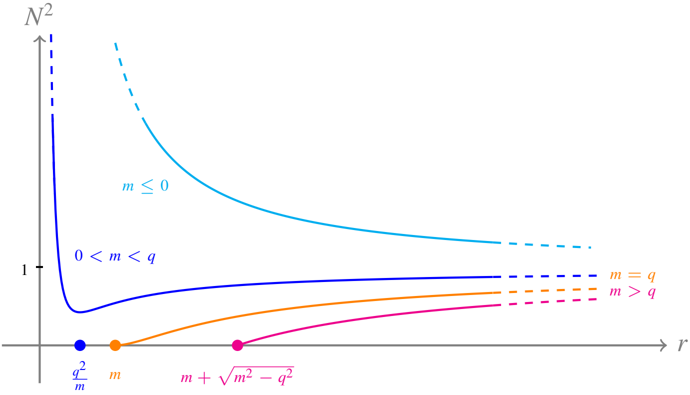
$$\mathrm{M} = [r_0, +\infty) \times \mathbb{S}^2 , \qquad g_0 = \frac{1}{N^2}\, \mathrm{d}r^2 + r^2\, g_{\mathbb{S}^2} ,$$

where the *Lapse Function* and the *Electric Potential* are

$$N = \sqrt{1 - \frac{2m}{r} + \frac{q^2}{r^2}} , \qquad \Psi = \frac{q}{r} .$$

The behaviour of the Reissner-Nordström solution and the interval $[r_0, +\infty)$ depend on the two parameters m and q:

- The *sub–extremal* case, when $m > q$
- The *extremal* case, when $m = q$
- The *super–extremal* case, when $m < q$



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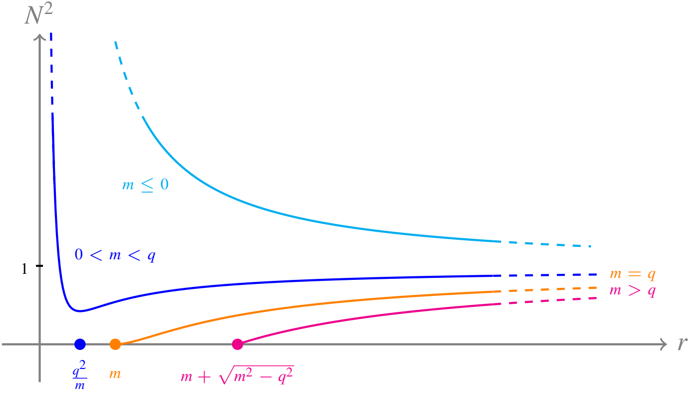

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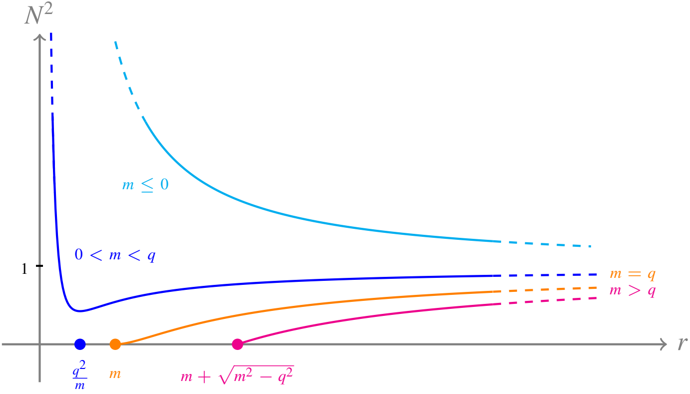

Uniqueness Theorems

Black Hole Uniqueness (Borghini, Cederbaum, C.)

Let $(\mathrm{M}, g_0, N, \Psi)$ be an asymptotically flat solution to (1).
Suppose that $\partial\mathrm{M}$ is connected and $N_0 = 0$. Then

- if the horizon $\partial\mathrm{M}$ is *non–degenerate* ($|\mathrm{D}N| \neq 0$ on $\partial\mathrm{M}$), then $(\mathrm{M}, g_0, N, \Psi)$ is isometric to a *sub–extremal* Reissner-Nordström,
- if the horizon $\partial\mathrm{M}$ is *degenerate*, $(\mathrm{M}, g_0, N, \Psi)$ is isometric to an *extremal* Reissner-Nordström.

The most complete results in the literature are:

- Israel, 1968
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Photon surfaces

Definition

- An embedded **timelike** hypersurface $P^3 \hookrightarrow (\mathcal{L}^4, g)$ is called a *Photon Surface* if it is **null totally geodesic**.
- A photon surface P^n is called *Equipotential* if the lapse function N and the electric potential Ψ are constant along each connected component of each time slice $\Sigma^2(t) := P^3 \cap \mathrm{M}(t)$.

If N and Ψ are constant on each connected component, independently of the time slice, P^3 is a *Photon Sphere*.

In Reissner-Nordström $\left\{ r = \frac{3m \pm \sqrt{9m^2 - 8q^2}}{2} \right\}$.

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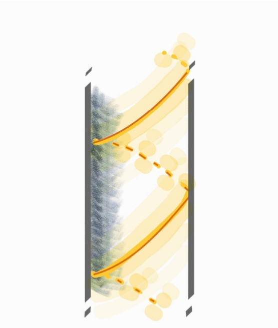

Properties of Equipotential Photon Surfaces

Let $\Sigma^2 = P^3 \cap \mathrm{M}$ be a compact time slice of an Equipotential Photon Surface with induced metric σ. Then $\left(\Sigma^2, \sigma\right)$ in $\left(\mathrm{M}, g_0\right)$

- is *totally umbilic*,
- its *mean curvature* H is constant,
- the *normal derivative* $\nu(N)$ is also constant on Σ^2 and

$$2\nu(N) = c\,\mathrm{H}\,N_0,$$

  for a constant $c \in \mathbb{R}$ (in the Photon Sphere case $c = 1$),
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Strategy of the proof

1. Introduce parameters $m, q \in \mathbb{R},\ q > 0$.

Consider

$$X = \Psi^2 + 1 - N^2 + \left(\frac{N_0^2}{\Psi_0} - \Psi_0 - \frac{1}{\Psi_0}\right)\Psi\,,$$

which satisfies

$$\Delta X \;=\; \frac{1}{N}\, g_0(\mathrm{D}X,\, \mathrm{D}N).$$

Since $X = 0$ on $\partial\mathrm{M}$ and $X \to 0$ at infinity,

$$N^2 \;=\; \Psi^2 - \frac{2m}{q}\Psi + 1 \qquad \text{on } \mathrm{M}\,.$$

Rewriting (1) in terms of the sole lapse function, the system is ill–posed on

$$\mathfrak{S} := \{N^2 + k = 0\}, \qquad k := \frac{m^2}{q^2} - 1\,.$$

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$$X = \Psi^2 + 1 - N^2 + \left(\frac{N_0^2}{\Psi_0} - \Psi_0 - \frac{1}{\Psi_0} \right) \Psi \,,$$

which satisfies

$$\Delta X \;=\; \frac{1}{N}\, g_0(\mathrm{D}X,\, \mathrm{D}N).$$

Since $X = 0$ on $\partial \mathrm{M}$ and $X \to 0$ at infinity,

$$N^2 \;=\; \Psi^2 - \frac{2m}{q}\, \Psi + 1 \qquad \text{on } \mathrm{M}\,.$$

Rewriting (1) in terms of the sole lapse function, the system is ill–posed on

$$\mathfrak{S} := \{N^2 + k = 0\}, \qquad k := \frac{m^2}{q^2} - 1\,.$$

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The following ideas are based on the cylindrical ansatz strategy introduced for the vacuum case by *V. Agostiniani and L. Mazzieri*, 2017.

2. Set the cylindrical ansatz.

Define the *pseudo–radial function* $\varrho : \mathrm{M} \to \mathbb{R}$ such that

$$N^2 = 1 - \frac{2m}{\varrho} + \frac{q^2}{\varrho^2},$$

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$$\varrho_\pm = \frac{q^2}{m \pm q\sqrt{N^2 + k}}, \qquad k := \frac{m^2}{q^2} - 1.$$

After chosing ϱ_+ or ϱ_-, introduce the *pseudo-affine function* $\varphi_\pm : M \to \mathbb{R}$,

$$\varphi_\pm = \log\left[\varrho_\pm(1 + N) - m\right].$$

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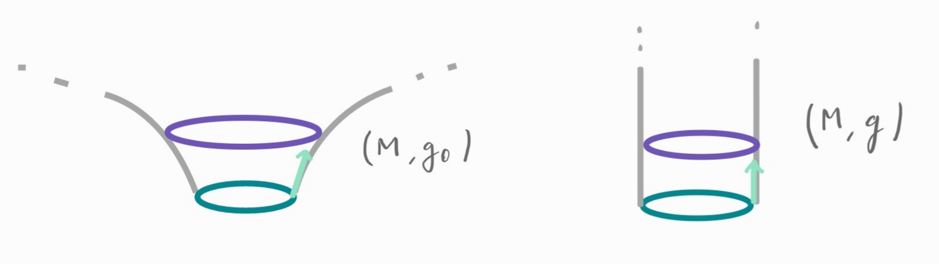
Cylindrical ansatz for Reissner-Nordström



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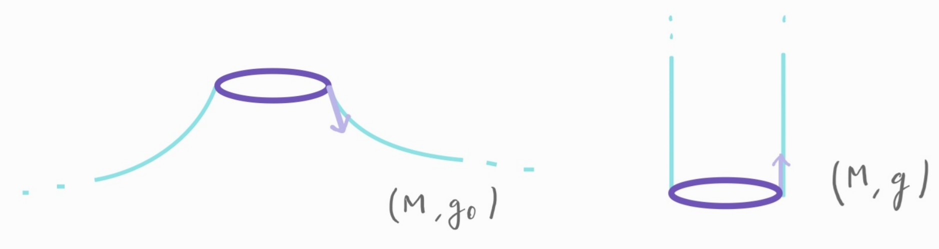

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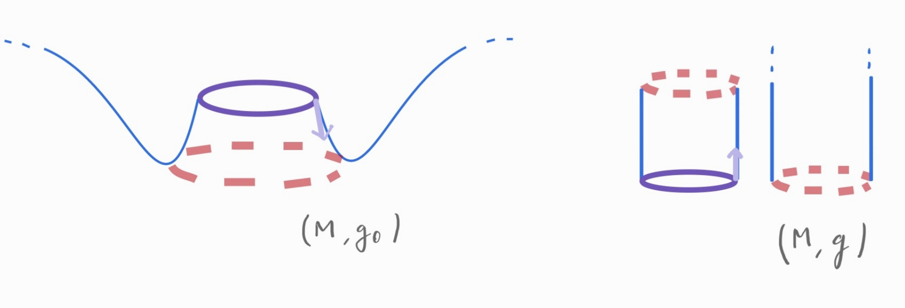

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$$\Uparrow$$

the conformal metric g is **cylindrical**

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$$|\nabla^2\varphi|_g \equiv 0 \quad \text{in M.}$$

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