Twistor geometry, non-linear structures, and perturbation theory

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- Hidden symmetries: objects more general than isometries: Killing tensors, Killing-Yano tensors, Killing spinors
- Teukolsky equations: perturbations reduce to a single scalar equation
- Reconstructions: symmetry operators map solutions of Teukolsky eqs. to linearized metrics (Hertz potentials)
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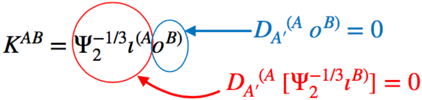

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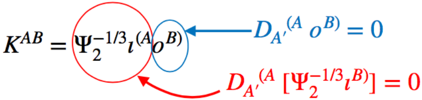

Question: What is the geometry underlying BH perturbation theory?

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- Riemannian version [Atiyah-Hitchin-Singer '78]: 'twistor space' is the space of complex structures

$$\left\{ \begin{matrix} \text{orthogonal almost} \\ \text{complex structures} \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{maximal isotropic} \\ \text{subspaces of } TM \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{projective} \\ \text{pure spinors} \end{matrix} \right\}$$

Remark: we allow different signatures and complex metrics.

- An almost-complex structure is a $(1,1)$ tensor J such that $J^2 = -1$, $J^{\mathrm{t}} g J = g$. It is equivalent to two projective spinors [BA '21a]:

$$J^a{}_b = \frac{i}{(o_C \iota^C)} (o^A \iota_B + \iota^A o_B) \delta^{A'}_{B'}$$

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- Gauge freedom:

conformal transf. $+$ rescalings of spinors

- This defines a 'gauge group' G_o. Fields transforming under G_o are sections of vector bundles E.

The complex-conformal connection

Theorem [BA '20, BA '21a]

- J induces a natural connection $\mathcal{C}_a = \mathcal{C}_{AA'}$ on E (covariant under conformal and projective transformations)
- J is half-integrable iff $\mathcal{C}_a o^B = 0$ or $\mathcal{C}_a \iota^B = 0$, and integrable iff both of these hold
- Let $\tilde{\mathcal{C}}_{A'} := o^A \mathcal{C}_{AA'}$ (partial connection). If $\mathcal{C}_a o^B = 0$ and Weyl is algebraically special, then $[\tilde{\mathcal{C}}_{A'}, \tilde{\mathcal{C}}_{B'}] = 0$.

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Remarks:

- Construction of $\mathcal{C}_a$: combine Lee form of J with 'GHP' connection
- Integrability is encoded in (non-linear) parallel spinors
- $[\tilde{\mathcal{C}}_{A'}, \tilde{\mathcal{C}}_{B'}] = 0 \Rightarrow$ 'flat connection' $\Rightarrow$ de Rham complex & parallel frames

Remark

The condition $\mathcal{C}_{AA'}o^B = 0$ is not only conceptually clear but also very useful in practice.

(To illustrate this, work out the simpler example $\nabla_{AA'}o^B = 0$)

Integration

- The condition $[\tilde{L}, \tilde{L}] \subset \tilde{L}$ gives $\tilde{L}$ the structure of a Lie algebroid $\Rightarrow \exists$ natural de Rham complex $(\Lambda^\bullet = \wedge^\bullet \tilde{L}, \, \tilde{\mathrm{d}})$:

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- We need forms with values on E. The connection $\mathcal{C}$ induces $\tilde{\mathrm{d}}^{\mathcal{C}}$.
- If Weyl= alg. special, then $(\tilde{\mathrm{d}}^{\mathcal{C}})^2 = 0$ and $(\Lambda^\bullet \otimes E, \tilde{\mathrm{d}}^{\mathcal{C}})$ is locally exact as well.

(In practice: if $\tilde{\mathcal{C}}^{A'}\varphi_{A'\ldots} = 0$, then $\varphi_{A'\ldots} = \tilde{\mathcal{C}}_{A'}\psi_{\ldots}$)

The conformal Einstein equations

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- Conformal structure $(M, [g])$ equipped with Weyl connection: $\nabla^{\rm w} g = 2{\rm w} \otimes g$, with ${\rm w} = {\rm d} \log \mathring{\Omega}$.
- The field equations are

$$\mathrm{Ric}^{\rm w} = \lambda g$$

- Reduction to ordinary Einstein: break conformal invariance $\mathring{\Omega} \equiv 1$

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- Φ satisfies the "conformal hyper-heavenly (CHH) equation"

$$(\mathcal{C}^a\mathcal{C}_a - 18\Psi_2)\Phi + \mathring{\Omega}(\tilde{\mathcal{C}}_{A'}\tilde{\mathcal{C}}_{B'}\Phi)(\tilde{\mathcal{C}}^{A'}\tilde{\mathcal{C}}^{B'}\Phi) - 4(\tilde{\mathcal{C}}^{A'}\mathring{\Omega})(\tilde{\mathcal{C}}^{B'}\Phi)(\tilde{\mathcal{C}}_{A'}\tilde{\mathcal{C}}_{B'}\Phi) = K$$

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- The linear term

$$(\mathcal{C}^a\mathcal{C}_a - 18\Psi_2)\Phi = 0$$

  is the Teukolsky equation.

Summary of key points:

- A **choice of complex structure** J determines conformally invariant connection
- Integrability of J encoded in (non-linear) **parallel spinors**. 'Hidden symmetries' are a consequence of this
- **Reduction** of (full, non-linear) conformal Einstein eqs. to CHH eq., and **reconstruction** of conformal structure
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Thanks!

References

- D. Bini, C. Cherubini, R. T. Jantzen and R. J. Ruffini, Prog. Theor. Phys. **107** (2002)
- M. Walker and R. Penrose, Commun. Math. Phys. **18** (1970), 265-274
- B. Araneda, Class. Quant. Grav. **35** (2018) no.7, 075015
- R. Penrose, Gen. Rel. Grav. **7** (1976), 31-52
- M. F. Atiyah, N. J. Hitchin and I. M. Singer, Proc. Roy. Soc. Lond. A **362** (1978)
- B. Araneda, arXiv:2106.01094
- B. Araneda, Lett. Math. Phys. 110, no. 10, 2603-2638 (2020)
- B. Araneda, arXiv:2110.06167
- J. F. Plebanski and I. Robinson, Phys. Rev. Lett. **37** (1976), 493-495