Sharp asymptotics for Teukolsky master equation on Kerr spacetimes

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Aim and Motivation

Aim: to prove the precise late time asymptotics for TME, satisfied by the extreme Newman–Penrose components of the spin-$|s|$ field, in the Kerr exterior:

$$0 = \Sigma\Box_g\psi_s + \frac{2is\cos\theta}{\sin^2\theta}\partial_\phi\psi_s - (s^2\cot^2\theta + s)\psi_s - 2ias\cos\theta\partial_t\psi_s$$
$$-2s\frac{r^3 - 3Mr^2 + a^2r + a^2M}{\Delta}\partial_t\psi_s - 2s(r-M)\partial_r\psi_s + 2s\frac{a(r-M)}{\Delta}\partial_\phi\psi_s.$$

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Motivation for asymptotics for TME

1. It determines the dynamics of the scalar field, Dirac field, Maxwell field and linearized gravity.
2. Sufficient *upper* bound of decay is essential for **Kerr stability conjecture**.
3. Appropriate *lower* bound of decay is indispensable for **Strong Cosmic Censorship conjecture** concerning the instability of the Kerr Cauchy horizon.

Late time tails for scalar wave ($s = 0$) on Schwarzschild

Scalar wave eq $-\mu^{-1}\partial_t^2\psi + r^{-2}\partial_r(\mu r^2\partial_r\psi) + r^{-2}\Delta_{S^2}\psi = 0$, $\mu = 1 - \frac{2M}{r}$.

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Backscattering due to an effective curvature potential: for $\Psi = r\psi$

$$-\mu^{-1}\partial_u\partial_v\Psi + \frac{\Delta_{S^2}\Psi}{r^2} - \frac{2M}{r^3}\Psi = 0,$$

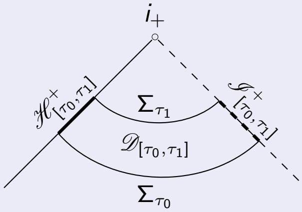
where $\partial_u = \partial_t - \mu\partial_r$ and $\partial_v = \partial_t + \mu\partial_r$ with

$$\mathrm{d}u = \frac{1}{2}(\mathrm{d}t - \mu^{-1}\mathrm{d}r), \mathrm{d}v = \frac{1}{2}(\mathrm{d}t + \mu^{-1}\mathrm{d}r).$$

Idea of proving precise asymptotics for $\ell = 0$ mode

Fix a hyperboloidal foliation (ρ, τ, ω), and assume some pointwise decay estimate.

Global conservation law for integral of radiation field along $\mathscr{I}^+_{[\tau_0,\infty)}$



Scalar wave eq is $\partial_\rho(\mu r^2 \partial_\rho \psi) = \partial_\tau H[\psi]$.
Integrate in $\mathscr{D}_{[\tau_0,\infty)}$,

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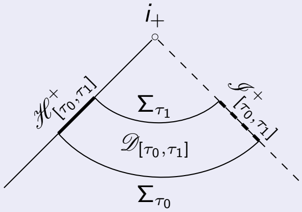

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Remark

This computes the integral of radiation field along $\mathscr{I}^+_{[\tau_0,\infty)}$ in terms of initial data.

Precise asymptotics of the $\ell = 0$ mode of Ψ

Recall $\mu^{-1}\partial_u\partial_v\Psi = -2Mr^{-3}\Psi$.

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\begin{aligned}
&v^3\partial_v\Psi(u,v) - v^3\partial_v\Psi(u_{\Sigma_{\tau_0}}(v), v)\\
&= -\int_{u_{\Sigma_{\tau_0}}(v)}^{u} M\mu\frac{v^3}{r^3}\Psi \mathrm{d}u\\
&= -M\int_{\mathscr{I}^+_{[\tau_0,\tau(u,\infty)]}} \Psi + l.o.t.\\
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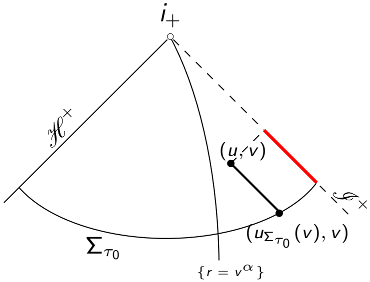

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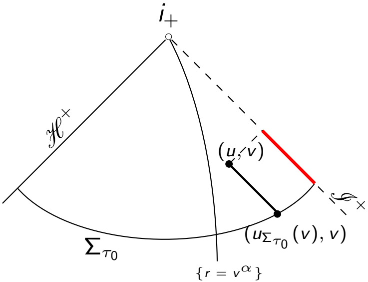

$\rightsquigarrow$ precise asymptotics of $\partial_v\Psi(u,v)$ in terms of

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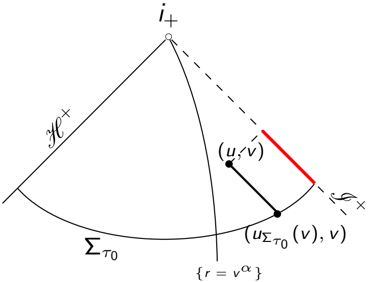

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$\rightsquigarrow$ precise late time asymptotics of Ψ in terms of initial data.

Price's law formulation of sharp decay for TME

As *both* upper *and* lower bounds for the decay rates

M.–Zhang (arXiv: 2108.03148, 2111.04489): $\mathfrak{s} = 0, 1, 2$

Schw.	towards null infinity	finite radius region
$r^{-\mathfrak{s}-s}\psi_s$	$r^{-1-\mathfrak{s}-s}\tau^{-2-\mathfrak{s}+s}$	$v^{-3-2\mathfrak{s}}$
$(r^{-\mathfrak{s}-s}\psi_s)_{\geq\ell}$	$r^{-1-\mathfrak{s}-s}\tau^{-2-\ell+s}$	$v^{-3-2\ell}$
total power	$-3-\mathfrak{s}-\ell$	$-3-2\ell$

Kerr	towards null infinity	finite radius region
$r^{-\mathfrak{s}-s}\psi_s$	$r^{-1-\mathfrak{s}-s}\tau^{-2-\mathfrak{s}+s}$	$v^{-3-2\mathfrak{s}}$
total power	$-3-2\mathfrak{s}$	$-3-2\mathfrak{s}$

Additionally, on $\mathscr{H}^+$, if $\mathfrak{s} \neq 0$ and $am = 0$, the decay of $\psi_{+\mathfrak{s}}$ is faster by v^{-1}.
(See **Barack–Ori '99**, but fails for $\mathfrak{s} = \frac{1}{2}$ by arXiv:2008.11429).
The coefficients in the global late time asymptotics are generically nonzero.

Ingredients of the proof

- We use spin-weighted spherical harmonic mode decomposition. For Kerr, because of $a^2 \sin^2\theta\partial_\tau^2 + ias\cos\theta\partial_\tau$ part in TME operator, one has to treat the mode coupling in the evolution. (Estimates for modes are coupled.)
- Extension of the approach in **M. '17, Andersson–Bäckdahl–Blue–M. '19** on Kerr linear stability.
- Global conservation law for null infinity integral of radiation field is crucial.
- Application of Teukolsky–Starobinsky identities is fundamental:

$$(\mathring{\eth}' - ia\sin\theta\partial_\tau)^{2\mathfrak{s}}\psi_{+\mathfrak{s}} \approx \Delta^{\mathfrak{s}}(\mu^{-1}V)^{2\mathfrak{s}}(\Delta^{\mathfrak{s}}\psi_{-\mathfrak{s}}),$$
$$(\mathring{\eth} + ia\sin\theta\partial_\tau)^{2\mathfrak{s}}\psi_{-\mathfrak{s}} \approx Y^{2\mathfrak{s}}\psi_{+\mathfrak{s}}.$$

Applications to the precise asymptotics

1. Higher modes of solutions to TME, see **Csukás–Rácz–Tóth '19** etc.
2. Semilinear problems, e.g., $\Box_g \psi = \pm \psi^k$ or satisfying null condition, see **Bizón–Chmaj–Rostworowski–Zajac '07–'09**, **Tohaneanu '21**.
3. Quasilinear problems.
4. Strong Cosmic Censorship.

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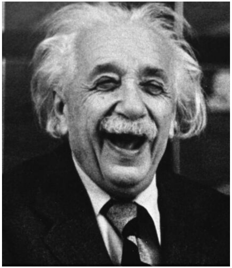

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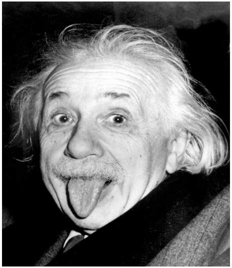

Asymptotics for TME on Schwarzschild

M.–Zhang (arXiv: 2108.03148): $\mathfrak{s} = 0, 1, 2$

- Let ψ_s be the spin s components on Schwarzschild satisfying TME. Then,

$$
\begin{aligned}
|r^{-2\mathfrak{s}}\psi_{+\mathfrak{s}} - \mu^{\mathfrak{s}}\frac{v+(2\mathfrak{s}+1)\tau}{v^{2\mathfrak{s}+2}\tau^2}\mathrm{Q}_{\mathrm{init}}| &\lesssim v^{-2\mathfrak{s}-1}\tau^{-2-\varepsilon} \\
|\psi_{-\mathfrak{s}} - \frac{(2\mathfrak{s}+1)v+\tau}{v^2\tau^{2\mathfrak{s}+2}}\mathrm{Q}_{\mathrm{init}}| &\lesssim v^{-1}\tau^{-2\mathfrak{s}-2-\varepsilon},
\end{aligned}
\tag{2}
$$

  and extra τ^{-1} for $\psi_{+\mathfrak{s}}$ on $\mathscr{H}^+$ if $\mathfrak{s} \neq 0$.
- If supported on $\geq \ell$ modes, then extra $C_{\mathfrak{s},\ell}h_{\mathfrak{s},\ell}r^{\ell-\mathfrak{s}}\tau^{-\ell+\mathfrak{s}}$ decay, $h_{\mathfrak{s},\ell}r^{\ell-\mathfrak{s}}$ a zero energy mode solution to TME of the ℓ mode of $\psi_{-\mathfrak{s}}$.

Asymptotics for TME on Kerr

M.–Zhang (arXiv: 2111.04489): $\mathfrak{s} = 0, 1, 2$

Let ψ_s be the spin s components on Kerr satisfying TME. Assume a Morawetz estimate for TME on subextreme Kerr. Then,

$$\begin{aligned} \Big|(r^2+a^2)^{-\mathfrak{s}}\psi_{+\mathfrak{s}} - \frac{v+(2\mathfrak{s}+1)\tau}{v^{2\mathfrak{s}+2}\tau^2}\sum_{-\mathfrak{s}\leq m\leq \mathfrak{s}} f_m \mathrm{Q}_{\mathrm{init},m}\Big| &\lesssim v^{-2\mathfrak{s}-1}\tau^{-2-\varepsilon} \\ \Big|\psi_{-\mathfrak{s}} - \frac{(2\mathfrak{s}+1)v+\tau}{v^2\tau^{2\mathfrak{s}+2}}\mathrm{Q}_{\mathrm{init}}\Big| &\lesssim v^{-1}\tau^{-2\mathfrak{s}-2-\varepsilon}, \end{aligned} \tag{3}$$

and extra τ^{-1} for $\psi_{+\mathfrak{s}}$ on $\mathscr{H}^+$ if $\mathfrak{s}\neq 0$ and $am = 0$.
Here, $f_m = \mu^{\mathfrak{s}} + amO(r^{-1})$, and $\mathrm{Q}_{\mathrm{init},m}$ and $\mathrm{Q}_{\mathrm{init}}$ are calculated from initial data.