Optimized coordinates for Ricci-flat conifolds

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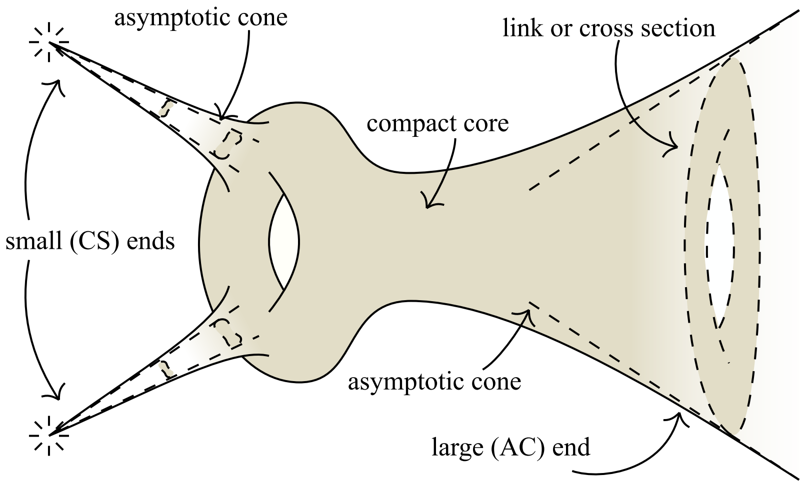
The star of the hour: Riemannian conifolds



Some history (without claim for completeness)

- gravitational instantons have been introduced in the 1970's, e.g. [EH78]
- Bartnik proved a positive mass theorem for AE manifolds [Bar86]
- Kronheimer classified four dimensional Ricci-flat ALE manifolds [Kro89]
- Bando, Kasue and Nakajima constructed coordinates at infinity [BKN89]
- several classes of examples have been constructed, even with special geometry e.g. [CH14]

Conifolds are also interesting in the study of the Ricci flow

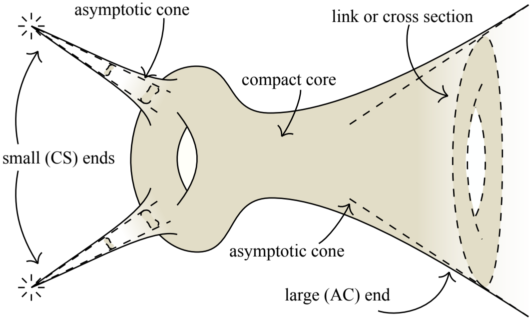
- Hamilton introduces the Ricci flow in 1982 $\rightsquigarrow$ conical singularities
- Nonlinear stability results for ALE manifolds based on optimized coordinates [DK20]

Goal: find a way to extend the results of [DK20].

Definition of a conifold

- A smooth manifold with ends is a manifold M such that $M = K \cup E_1 \cup \ldots \cup E_m$ where $K \subset M$ is compact and $E_j \simeq \mathbb{R} \times N_j$ as manifolds.
- Given a Riemannian manifold with ends, an end E_j is called
  - an asymptotically conical (AC) end if there is a diffeomorphism $\phi_j \colon E_j \to (R, \infty) \times N_j$ with
  $$|\nabla^k(\phi_* g - g_{\text{cone}})| = \mathcal{O}(r^{-\tau_j - k})$$
  for all $k \in \mathbb{N}$ as $r \to \infty$,
  - conically singular (CS) if there is a diffeomorphism $\phi_j \colon E_j \to (0, R) \times N_j$ with
  $$|\nabla^k(\phi_* g - g_{\text{cone}})| = \mathcal{O}(r^{+\tau_j - k})$$
  for all $k \in \mathbb{N}$ as $r \to 0$,

  where $g_{\text{cone}} = dr \otimes dr + r^2 g_{N_j}$ is the cone metric.
- A conifold is a Riemannian manifold with ends if each of its ends is either AC or CS.



PDE technology on conifolds

The usual techniques of PDE theory, like

- Sobolev and Hölder spaces
- various embedding theorems
- elliptic estimates,

can be extended to this setting by introducing weighted norms [Can75, Can79, LM85, Bar86, Pac13, Bam11].

$$\|u\|_{L^2_\beta} = \left(\int_M |\rho^{-\beta} u|^2 \rho^{-n} d\mu\right)^{1/2} \qquad \|u\|_{H^k_\beta} = \sum_{l=0}^{k} \|\nabla^l u\|_{L^2_{\beta-l}}$$

$$\|u\|_{C^{k,\alpha}_\beta} = \sum_{l=0}^{k} \sup_{x\in M} \rho^{-\beta+l}(x)|\nabla^l u(x)|$$
$$+ \sup_{\substack{x,y\in M\\ 0<d(x,y)<\mathrm{inj}(M)}} \min\left\{\rho^{-\beta+k+\alpha}(x), \rho^{-\beta+k+\alpha}(y)\right\} \frac{|\tau_x^y \nabla^k u(x) - \nabla^k u(y)|}{d(x,y)^\alpha},$$

Gauging

We are interested in Ricci-flat manifolds.

$$\mathrm{Ric}(g) = 0$$

Diffeomorphism-invariance $\rightsquigarrow$ degenerate symbol $\rightsquigarrow$ inconvenient to work with.
Solution: introduce a term that "counteracts the diffeomorphism action".
Fix a background metric $\tilde{g}$, and consider the **Ricci–DeTurck** PDE [DeT83, AM03]

$$-2\,\mathrm{Ric}(g) + \mathcal{L}_{V(g,\tilde{g})} g = 0,$$

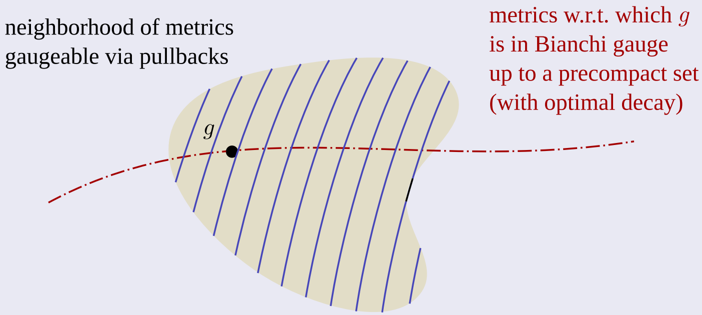
where $V(g,\tilde{g}) := g^{-1} \circ (\nabla^g - \nabla^{\tilde{g}}) = g^{ij}(\Gamma(g)_{ij}{}^k - \Gamma(\tilde{g})_{ij}{}^k)\partial_k$, is an **elliptic quasi-linear PDE**.

Definition 1 (Bianchi, or harmonic, gauge)

A metric g is in Bianchi gauge with respect to $\tilde{g}$ if the vector field $V(g,\tilde{g})$ vanishes everywhere, except possibly on a precompact set (cf. CMCSH gauge in Zoe Wyatt's lecture).

Theorem 2 (local slice theorem, Kröncke–ÁS [KS])

The Bianchi gauge provides a good local slice for the diffeomorphism action on metrics. That is, given a background metric $\tilde{g}$ and a precompact set U, there is a neighbourhood of $\tilde{g}$ in a suitable weighted Sobolev space such that any metric in this neighbourhood can be pulled back by a unique diffeomorphism (close to the identity) to a metric which is in Bianchi gauge everywhere except possibly on $\overline{U}$.



The linearized problem

- The linearization of the Ricci–DeTurck operator at a Ricci-flat metric on the diagonal is

$$\left.\frac{d}{dt}\right|_{t=0} \left(-2\,\mathrm{Ric}(g+th) + \mathcal{L}_{V(g+th,g)}g\right) = \nabla^{g*}\nabla^g h + h \circ \mathrm{Ric}^g - \mathrm{Ric}^g \circ h - 2\mathring{R}^g h =: \Delta_L^g(h),$$

  where the last term is of order zero and depends on the curvature.
- On a cone

$$\Delta_L = -\nabla_{\partial_r} \circ \nabla_{\partial_r} - \frac{n-1}{r}\nabla_{\partial_r} + \frac{1}{r^2}\Box_L,$$

  where $\Box_L$, the tangential operator, is an r-independent second-order operator containing no radial derivatives.
- The Laplace–Beltrami operator and the Hodge Laplacian have similar decompositions. Thus we obtain the tangential operators $\Box_0 = \Delta^{\mathrm{cone}}$ and $\Box_1$.

The spectrum of the tangential operator on a cone

Theorem 3 (Kröncke–ÁS [KS])

The spectrum of the tangential operator $\square_L$ of the Lichnerowicz Laplacian is given by

$$\begin{aligned}\sigma(\square_L) = \ &\sigma(\Delta_L^{link}|_{TT}) && \textit{from transverse traceless tensors}\\ &\cup \left\{F_\pm(\mu)\ \middle|\ \mu \in \sigma(\Delta_1|_{D(\mathrm{link})})\right\} && \textit{from divergence free 1-forms}\\ &\cup \sigma(\Delta_B^{link}) \cup \left\{G_\pm(\lambda)\ \middle|\ \lambda \in \sigma(\Delta_B^{link})\right\} && \textit{from functions}\\ &\cup \{0, 2\dim M - 2\},\end{aligned}$$

where $F_\pm$ and $G_\pm$ are concretely given elementary functions.

Simplifying assumption

For ease of presentation, we will assume from now on that the critical value $-\left(\frac{n-2}{2}\right)^2$ is not in the spectrum of the tangential operator $\square_L$.

The relation between the spectrum and decay rates

For a 2-tensor $h = f(r)r^2k(x)$ in product form where $\Box k = \nu k$, then $\nabla_{\partial_r}(r^2k) = 0$ and we have
$\Delta_L = -\nabla_{\partial_r} \circ \nabla_{\partial_r} - \frac{n-1}{r}\nabla_{\partial_r} + \frac{1}{r^2}\Box$

$$\begin{aligned} \Delta_L h &= -f''(r)r^2k - \frac{\dim M - 1}{r} f'(r)r^2k + \frac{\nu}{r^2} f r^2 k \\ &= \left(-f''(r) - \frac{\dim M - 1}{r} f'(r) + \frac{\nu}{r^2} f \right) r^2 k \end{aligned}$$

$\rightsquigarrow$ the decay rate in the kernel of Δ_L is determined by the spectrum of the tangential operator $\Box$.
The resulting ODE

$$-f''(r) - \frac{\dim M - 1}{r} f'(r) + \frac{\nu}{r^2} f = 0$$

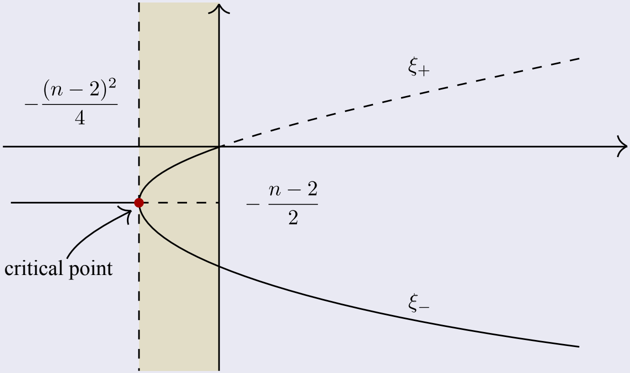
can be solved explicitly: $f(r) = c_+ r^{\xi_+(\nu)} + c_- r^{\xi_-(\nu)}$, where $\xi_\pm(\nu) = -\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \nu}$ are the indicial roots corresponding to ν.

Decay rates

Theorem 4 (Decay in the kernel of the Lichnerowicz Laplacian, Kröncke–ÁS [KS])

A tensor field h which decays at infinity and which satisfies $\Delta_L^{cone} h = 0$ has $h = O\left(r^{-\xi}\right)$ where $\xi := \min\{-\Re\xi_\pm(\nu) \,|\nu \in \sigma(\Box_L)\} \cap (0,\infty)$.

$$\xi_\pm(\nu) = -\tfrac{n-2}{2} \pm \sqrt{\left(\tfrac{n-2}{4}\right)^2 + \nu}$$



Proposition 5 (Kröncke–ÁS [KS])

Let $(\overline{M}, \overline{g})$ be a Ricci-flat cone and let g be a Ricci-flat metric defined on an open set $U \subset \overline{M}$ which is in Bianchi gauge with respect to $\overline{g}$. Then $g - \overline{g} = \mathcal{O}_\infty(r^{-\xi})$ as $r \to \infty$.

Idea of the proof.

$$2\,\mathrm{Ric}(g) = \mathcal{L}_{V(g,\overline{g})}\overline{g} \qquad \text{on } U.$$

Due to [Shi89, Lemma 2.1], this can be rewritten in terms to the difference $h = g - \overline{g}$ as

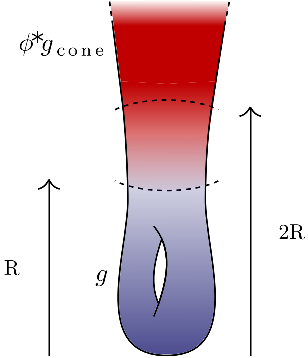
$$\overline{\Delta}_L h = g^{-1} * \overline{\mathrm{Rm}} * h * h + g^{-1} * g^{-1} * \overline{\nabla} h * \overline{\nabla} h + g^{-1} * \overline{\nabla}^2 h * h. \tag{1}$$

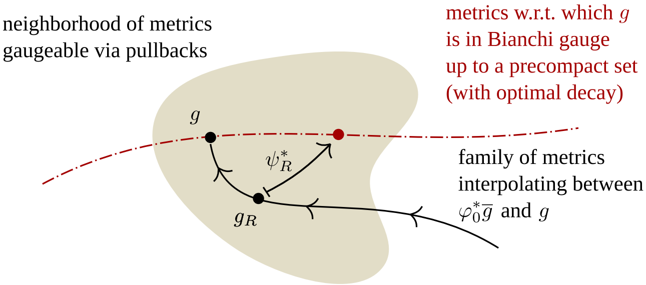
Note that the RHS is quadratic in h. The rest of the proof is a standard iteration procedure in weighted function spaces.

Optimalized coordinates

Theorem 6 (Kröncke–ÁS [KS])

Let (M, g) be an asymptotically conical Ricci-flat manifold. Then there exist compact set $K \subset M$ and an asymptotic chart $\varphi : M \setminus K \to \overline{M}_{>R}$ such that $\varphi_* g - \overline{g} \in \mathcal{O}_\infty(r^{-\xi})$ as $r \to \infty$.





Take-home message

Conifolds care a lot about their ends

Even though conifolds have a relatively large number of degrees of freedom, the condition of Ricci-flatness brings so much ridigity into the picture that the spectra of operators on the link of the ends to determine much about the decay.

Thank you for your attention!

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Theorem 7 (Kröncke–ÁS [KS])

Let $(\overline{M}^n, \overline{g})$ *be a Ricci-flat cone over a closed manifold* $(\widehat{M}^{n-1}, g)$ *with* $\widehat{\mathrm{Ric}} = (n-2)\widehat{g}$.
Let $0 = \lambda_0 < \lambda_1 \ldots$ *be the eigenvalues of the Laplace–Beltrami operator on* $\widehat{M}$, $\mu_1 < \mu_2 < \ldots$ *be the eigenvalues of the connection Laplacian on divergence-free 1-forms on* $\widehat{M}$ *and* $\kappa_1 < \kappa_2 < \ldots$ *be the eigenvalues of the Einstein operator on transverse and traceless tensors on* $\widehat{M}$.

(i) *The indicial set of the Lichnerowicz Laplacian* $\overline{\Delta}_L$ *on* $\overline{M}$ *is given by*

$$\{\xi_\pm(\kappa_i), \xi_\pm(\mu_i+1)-1, \xi_\pm(\mu_i+1)+1, \xi_\pm(\lambda_i)-2, \xi_\pm(\lambda_i), \xi_\pm(\lambda_i)+2 \,|i \in \mathbb{N}\} \cup \{-n, 2-n, 0, 2\}.$$

(ii) *The indicial set of* $\overline{\Delta}_L$ *on tensors satisfying the linearized Bianchi gauge is given by*

$$\{\xi_\pm(\kappa_i), \xi_\pm(\mu_i+1)-1, \xi_\pm(\lambda_i)-2, \xi_\pm(\lambda_i) \,|i \in \mathbb{N}\} \cup \{-n, 0\}.$$

(iii) *The indicial set of* $\overline{\Delta}_L$ *on tensors satisfying the linearized Bianchi gauge, but which are not Lie derivatives, is given by* $E := \{\xi_\pm(\kappa_i), \xi_\pm(\lambda_i) \,|i \in \mathbb{N}\}$.

$\xi_\pm(x) := -\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + x}$.

$$
\begin{aligned}
E_+ &:= \mathrm{Re}(E) \cap (0,\infty) = \{\xi_+(\kappa_i), \xi_+(\lambda_i) \,|i \in \mathbb{N}, \kappa_i > 0\} \\
E_- &:= \mathrm{Re}(-E) \cap (0,\infty) \\
&= \{-\xi_-(\kappa_i), -\xi_-(\lambda_i) \,|i \in \mathbb{N}\} \cup \left\{ -\xi_+(\kappa_j) \,\middle|\, i \in \mathbb{N}, -\frac{(n-2)^2}{4} \leq \kappa_j < 0 \right\} \\
&\quad \cup \left\{ -\mathrm{Re}(\xi_\pm(\kappa_j)) = \frac{n-2}{2} \,\middle|\, i \in \mathbb{N}, \kappa_j < -\frac{(n-2)^2}{4} \right\}, \\
\xi_+ &:= \min E_+ \qquad \xi_- := \min E_-.
\end{aligned}
$$

Theorem 8

Kröncke–ÁS [KS] Let (M^n, g) be a Ricci-flat conifold with ends M_i, $i = 1, \dots, N$, which are modeled by Ricci-flat cones over Einstein manifolds $(\widehat{M}_i, \widehat{g}_i)$. Then the following assertions hold:

(i) *If M_i, $i \in \{1, \dots, N\}$, is an asymptotically conical end, then it is of order $\xi_-(\widehat{M}_i, \widehat{g}_i)$ if it is not resonance-dominated and weakly of order $\frac{n-2}{2}$ otherwise.*

(ii) *If M_i, $i \in \{1, \dots, N\}$, is a conically singular end, then it is of order $\xi_+(\widehat{M}_i, \widehat{g}_i)$.*